Physics 253 B, Problem Set 3

November 26, 2019

(1) BCFW for NMHV Tree Amplitudes Using on-shell diagrams, work out the permutations for all terms in any way you like of doing BCFW for NMHV tree amplitudes. Illustrate in pictures what the 6 terms for $n = 7$ look like as we did for $n = 6$, i.e. draw the configuration of the 7 columns of the C matrix, projectively, as points in two dimensions. Show what the permutation and the associated plabic graph looks like in the momentum-twistor Grassmannian, again for $n = 7$. For all n, find the especially simple way of running the recursion where the expression for the amplitude turns out to be the one we have used in class, $\sum_{i<j}[1, i, i+1, j, j+1]$.

(2) Counting for BCFW trees Show that, for any n, the *total* number of terms (for all k) contributing to BCFW at tree-level are given by the Catalan numbers C_{n+3}, which satisfy the recursion relation

$$C_{n+1} = \sum_j C_j C_{n-j} \tag{1}$$

Show that the generating function for all the Catalan numbers, $C(x) = \sum_n C_n x^n$, satisfies $C(x) = 1 + xC(x)^2$ and use this to solve for $C(x)$ and get the explicit expression $C_n = \frac{1}{(n+1)} \begin{pmatrix} 2n \\ n \end{pmatrix}$. Show that for fixed k, the number of terms are given by the Narayana numbers $N_{n+3,k-1}$. Derive the recursion relation and the qudratic equation for the generating function of the Naryana numbers, to find $N_{n,k} = \frac{1}{n} \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} n \\ k-1 \end{pmatrix}$.

(3) On-shell diagram proof of BCFW Show, only using on-shell diagrams, that any way of running BCFW recursion at tree-level gives you the same result, i.e. that all the physical factorization channels are correctly captured and that all the spurious poles cancel in the sum. For the factorization channels, some (where the deformed particles are on opposite sides) are built-in to BCFW, so the interesting part is to show that all the rest of them are correctly captured. The diagrams for the spurious poles come in pairs and cancel. Flesh out the discussion given on pages 120-121 of https://arxiv.org/pdf/1212.5605.pdf.

(4) Top cell and canonical form for $G_+(2,n)$ Use bridge decomposition to explicitly find positive co-ordinates$z_A, A = 1, \cdots 2n-4$ for the top cell of $G_+(2,n)$, and give a formula for the minors (ij) in terms of the z_A. Verify that the co-ordinate invariant canonical form matches the product of all the dlogs of the positive co-ordinates, i.e.

$$\frac{d^{2\times n}C/GL(2)}{(12)(23)\cdots(n1)} = \prod_A \frac{dz_A}{z_A} \tag{2}$$

(5) Dual amplituhedron for NMHV Recall that the canonical form of a polytope is the volume of it's dual. We can use this picture to give new expressions for NMHV tree amplitudes, with only physical poles. To begin with, we can practice with the $m = 3$ amplituhedron. (This turns out to actually compute, for $k = 1$, a particular "split-helicity" component of the NMHV amplitude for $n + 1$ particles, where the three negative helicities are next to each other). First, show that the boundaries of the $m = 3$ amplituhedron for general n look like $(1ii+1)$ and $(ii+1n)$. Give a natural triangulation of this $m = 3$ cyclic polytope. Next, draw a picture of the polytope for $n = 5$, it has 5 vertices, 6 faces and 9 edges. The two-piece triangulation is obvious from the picture, adding one tetrahedron on top of another one. Next, draw a picture of the dual polytope, which has 6 vertices, 9 edges and 5 faces. As you might expect, this is what you get from starting with a tetrahedron and chopping off a corner. From the picture, give some natural triangulations of the dual polytope, and write down a the expression for it's volume and hence the canonical for of the original polytope. By construction, the expression will not have any spurious poles. Try and extend this picture for $m = 3$ to all n. Finally, see if you can find the triangulation of the dual of the cyclic polytope for $m = 4$, $n = 6$. You may find the discussion in https://arxiv.org/pdf/1012.6030.pdf, section 4, useful.

(6) Direct determination of the canonical form at 1-loop In class, we discussed how to directly determine the canonical form of a polygon, by writing down an expression with only the obvious poles corresponding to codimension one boundaries, and working out what the numerators had to be in order to put zeroes on the "wrong" residues where non-adjacent sides of the polygon meet. Here we will pursue this strategy to get the canonical form for 1-loop amplitudes which are equivalent to the canonical form for the $m = 2$ amplituhedron. In general we start with a form having poles $\langle Yii+1\rangle$. Just by weights there is some numerator of the form of weight $(n-4)$ in Y. You should find that there are certain triples i, j, k for which $\langle Yii+1\rangle, \langle Yjj+1\rangle, \langle Ykk+1\rangle \to 0$ are not boundaries of the amplituhedron, and the job of the numerator is to put zeroes there. Determine the numerator for $n = 5$ explicitly in this way.

(7) The $m = 2$ amplituhedron form in Z space In class we described a triangulation for the $m = 2$ amplituhedron, for all n and k. Find the canonical form $m \times k = 2 \times k$ form associated with this triangulation. Next, look at this canonical form thought of as being defined on the little z_a space– the two-dimensional space of the Z_a projected through Y. For $k = 1$, this form is

$$\Omega_{k=1}(z) = \sum_i \mathrm{dlog}\left(\frac{\langle 1i\rangle}{\langle ii+1\rangle}\right) \mathrm{dlog}\left(\frac{\langle 1i+1\rangle}{\langle ii+1\rangle}\right) \tag{3}$$

Remarkably, for any k, the canonical form in z space turns out to be

$$\Omega_k(z) = \left(\Omega_{k=1}(z)\right)^k \tag{4}$$

See if you can prove this, or at least show why it's not obviously wrong i.e. why the form doesn't have double-poles.

(8) Eight point N^{2}MHV and the amplituhedron Find expressions for the C matrices of all 20 terms for the $n = 8$, $k = 2$ tree amplitude. Using the computer, show that none of these cells are overlapping in the $Y = C.Z$ map. One way of doing this is as follows. Pick some positive co-ordinates for one cells A and put $Y_A = C_A \cdot Z$ in that cell. You should then be able to solve for the co-ordinates of another cell B i.e. by solving $Y_A = C_B \cdot Z$. You should then find that some of the B cell co-ordinates must be negative, and hence the A and B cells don't overlap.