Physics 253 B, Problem Set 2

October 15, 2019

(1) Cross Ratios Consider n points $X_1, X_2, \cdots, X_n$ in $\mathbb{P}^1$. We can think of these as 2-dimensional vectors, modulo the "torus" rescaling $X_i \to t_i X_i$, as well the action of $SL(2)$. So this space is $2 \times n - n - 3 = (n-3)$ dimensional.The $SL(2)$ invariance tells us all invariants we consider must be functions of the brackets $(ij) = \epsilon_{ab} X_i^a X_j^b$, and we can only work with combinations of these brackets for which the torus weight under rescaling each variable is zero. For $n = 3$, there are no degrees of freedom, and this is compatible with the fact that there are no torus invariants we can build. For $n = 4$, the space is one-dimensional, and correspondingly we have an invariant, the cross-ratio:

$$[12; 34] = \frac{(12)(34)}{(23)(41)} \tag{1}$$

Show that all the other combintations you can build of the form $[ab; cd]$ are simply related to the above. Do this both by playing with the Plucker relations, as well as by conveniently gauge-fixing the torus and $SL(2)$ action. For the latter argue that you can use $SL(2)$ as well as rescaling the columns to e.g. bring the four vectors $X1, \cdots, X4$ to the form $(1,0), (1,z), (1,1), (0,1)$ respectively. (We usually refer to this as setting 1,3,4 to 0,1 and infinity). Then compute all the different brackets in this gauge-fixing and verify their relationship to each other.

(2) The Riemann Sphere We most usually think of points on a 2-sphere as associated with a three-dimension unit vector $\hat{n}$. But we can also think of this as associated with a null 4-vector $p^\mu = (1, \hat{n})$, $p^2 = 0$. Then we can write as usual $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$, where we identify $\tilde{\lambda} = \lambda^*$ so that we have a real 4-momentum in minkowski space (and thus a real 2-sphere). This identification manifests the action of $SL(2, \mathbb{C})$ on the 2-sphere. Now, suppose we are given

4 points on the sphere; we can associated them with $\lambda_i, \tilde{\lambda}_i$ for $i = 1, \cdots, 4$. The invariant information in this configuration is encoded in the two cross-ratios

$$c = \frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 41\rangle}, \; \tilde{c} = \frac{[12][34]}{[23][41]} \tag{2}$$

Find expressions for $c + \tilde{c}$ and $c\tilde{c}$ in terms of the dot products $p_i \cdot p_j = (1 - \hat{n}_i \cdot \hat{n}_j)$, and write down the quadratic equation that has $c, \tilde{c}$ as it's two roots. Show that $c - \tilde{c}$ is proportioanl to the parity-odd object $\epsilon_{\mu_1 \cdots \mu_4} p_1^{\mu_1} \cdots p_4^{\mu_4}$. This shows that $c - \tilde{c} = 0$ if and only if the 4-vectors $p_1, \cdots, p_4$ are linearly dependent, which means that $\hat{n}_1, \cdots, \hat{n}_4$ lie on a plane, or that the 4 points on the sphere are "cocircular".

(3) Projective Practice

(A) Consider nine points $X1, \cdots, X9$ in $\mathbb{P}^3$. Determine the point on which the three planes $(123), (456), (789)$ intersect.

(B) Suppose we have a line (12), as well as planes (345) and (678) in $\mathbb{P}^2$. We want to determine the condition that the the line (12) intersects the line where the planes intersect $(345) \cap (678)$. Show that this requires

$$\langle (12) \cap (345) \cap (678) \rangle = \langle 1345\rangle\langle 2678\rangle - \langle 2345\rangle\langle 1678\rangle = 0 \tag{3}$$

(C) Consider a polygon with vertices $(X_1, X_2, X_3, X_4, X_5)$, and look at the line (51). There are four natural points on this line: of course X_5, X_1, but also the intersection of (23) with (51), and of (34) with (51). So, we have four points on the line (51). Express the cross-ratio of the four-points on this line in terms of the 3-brackets made from the X's.

(D) Prove the Pappus theorem by brute force. Recall the statement: take three points on a line $(1, 2, 3)$, and another three points on a line (A, B, C). Consider the points X where $(1B)$ intersects $(2A)$, Y where $(2C)$ intersects $(3B)$ and Z where $(3A)$ intersects $(1C)$. The claim is that X, Y, Z are collinear. Verify this by writing 3 as a linear combination of $(1, 2)$, C as a linear combination of (A, B) and grinding it out.

(4) Six Points on a Conic Just by counting parameters, we see that any five points in $\mathbb{P}^2$, $X_1^I, \cdots, X_5^I$, determine a conic C_{IJ}. So if we are given *six* points, they must satisfy a single relation for all six to lie on a conic. We wish to find that relation. In other words, we want to find the condition that needs to be satisfied in order for there to exist some C_{IJ} for which $C_{IJ} X_i^I X_i^J = 0$ for all $i = 1, \cdots, 6$. This has a simple geometric interpretation. For any X^I in $\mathbb{P}^2$, we associated some $x^{IJ} = X^I X^J$ which is symmetric in the IJ

indices.We can group all the indices (IJ) into a $3 \times 4/2 = 6$ vector, so x^{IJ} can be thought of as a point in $\mathbb{P}^5$. But then C_{IJ} is a hyperplane in $\mathbb{P}^5$. Thus we are looking for a condition for 6 points in $\mathbb{P}^5$ to lie on a plane. But that is simply the condition for the 6-bracket of $\langle x_1 \cdots x_6 \rangle = 0$, or more explicitly

$$S = \epsilon_{(I_1J_1)\cdots(I_6J_6)}(X_1^{I_1}X_1^{J_1}) \cdots (X_6^{I_6}X_6^{J_6}) = 0 \tag{4}$$

where $\epsilon_{(I_1J_1)\cdots(I_6J_6)}$ is symmetric in the (IJ) but totally antistymmetric under interchange of pairs $(I_aJ_a),(I_bJ_b)$. Argue this is some quartic expression in 3-brackets, of weight 2 in each of the X_i. There are many equivalent ways of expressing S explicitly in terms of 3-brackets; one which is easy to remember is

$$S = (123)(345)(561)(246) - (234)(456)(612)(351) \tag{5}$$

You can verify that this is correct again by direct computation. It is also illuminating to check that $S = 0$ for 6 points on a conic. Consider the conic curve which is just a parabola or more fancily "the moment curve" $X(t) = (1, t, t^2)$, and choose $X_i = X(t_i)$ for some set of six t_i. Show that the 3-bracket $(X_aX_bX_c)$ is the vandermonde determinant $(abc) = (t_b - t_a)(t_c - t_b)(t_c - t_a)$, and use this to show easily that $S = 0$ for any six points on this curve.

(5) The Schubert Problem This is the first classic problem in enumerative geometry, and it crops up all over the place in the physics of scattering amplitudes and elsewhere. Supose you are given 4 lines L_1, L_2, L_3, L_4 in $\mathbb{P}^3$. How many lines L_* intersect all 4? Using the twistor correspondence, we can translate this to another question. Given four points in spacetime $x_1, \cdots, x_4$, find other points x_* that are null-separated from all four; how many are there? Argue from both points of view that there are two solutions for the line L_* intersecting $L_1, \cdots, L_4$/the point x_* null-separated from $(x_1, \cdots, x_4)$. (The solutions can in general be complex of course). To gain intuition for the case of lines, first imagine somewhat special configurations, where e.g. the lines L_1, L_2 intersect and the lines L_3, L_4 intersect. To gain intuition for the case of null separation from points, imagine to begin with that all four of the x's lie in Euclidean space; note that any 4 points in Euclidean space can be taken to lie on a 2-sphere, and take it from there.

Consider various degenerate cases more explicitly. To be concrete, consider seven points $Z_1, \cdots, Z_7$, and the associated seven lines $L_i = Z_iZ_{i+1}$. Take any four of these seven lines, (at least one pair of them will intersect). Classify all the different cases you can have, and in all these cases, explicitly determine the line that intersects all four.

Now return to the general problem. On any one of the lines L_*, there will be four points–the intersection of L_* with the lines $L_1, \cdots, L_4$. Compute the cross-ratio of the 4 points on this line. You should find a quadratic equation remarkably similiar to the one encountered in the Riemann sphere problem above.

(6) Explicit dlog representation of Canonical Forms

(A) Consider the canonical form of a triangle,

$$\Omega = \frac{\langle Y d^2 Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} \tag{6}$$

Show that this can be rewritten in a way that is manifestly a dlog form, as for instance

$$\Omega = dlog\left(\frac{\langle Y23\rangle}{\langle Y12\rangle}\right) dlog\left(\frac{\langle Y31\rangle}{\langle Y12\rangle}\right) \tag{7}$$

Do this again in two ways: first by playing with Plucker relations, but also by choosing co-ordinates (like $Y = Z_3 + y_2 Z_2 + y_1 Z_1$) and explicitly seeing the dlog form.

(B) Now consider the form corresponding to a general 1-loop "box" integrand:

$$\Omega = \frac{d^4 x N}{(x-x_1)^2 \cdots (x-x_4)^2} \tag{8}$$

where N is a normalization factor depending on $(x_1, \cdots, x_4)$ we will return to in a moment. We can equivalently think of this as a form on the space of lines (AB) in $\mathbb{P}^3$ as

$$\Omega = \frac{d^4 Z_A d^4 Z_B}{GL(2)} \frac{N}{\langle ABL_1\rangle \cdots \langle ABL_4\rangle} \tag{9}$$

Show that with an appropriate choice of normalization, Ω is a dlog dform! Indeed, we have that

$$\Omega = dlog\frac{(x-x_1)^2}{(x-x_*)^2} \cdots dlog\frac{(x-x_4)^2}{(x-x_*)^2} = dlog\frac{\langle ABL_1\rangle}{\langle ABL_*\rangle} \cdots dlog\frac{\langle ABL_4\rangle}{\langle ABL_*\rangle} \tag{10}$$

where L_*, x_* are the the solutions of the Schubert problem discussed above. Note that we can choose either of the two solutions for L_*/x_*, and the canonical form just changes by a sign.

You can have fun seeing how this works by playing with special cases. For instance suppose we have four points $Z_1, \cdots, Z_4$ and that $L_i = (Z_i Z_{i+1})$

as usual. We can gauge-fixed the $GL(2)$ by setting A to the point where the line (AB) intersects the plane (123) and B to where it intersects the plane (341), in equations, putting e.g. $A = Z_1 + xZ_2 + wZ_4, B = Z_3 + yZ_2 + zZ_4$. The numerator is just fixed to be $\langle 1234\rangle^2$ by weights, and you can easily see that $\Omega = dlogx\, dlogy\, dlogz\, dlogw$. Do the same analysis for a few of the other special cases where the lines intersect, and extend to the general case.

(7) Bosonized SUSY

Consider the expression for the 3-bracket [123] where we replace the grassmann variable η^I with the differential dZ^I, where $I = 1, 2$:

$$[123] = \frac{(\langle 12\rangle \eta_3 + cyclic)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \to \frac{(\langle 12\rangle dZ_3 + cyclic)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \tag{11}$$

Show that this 2-form is actually a dlog form:

$$dlog\frac{\langle 23\rangle}{\langle 12\rangle} dlog\frac{\langle 31\rangle}{\langle 12\rangle} \tag{12}$$

The result extends in the obvious way to m-forms $[12\cdots m+1]$, including the case of the BCFW terms for $N = 4SYM$ where we have $m = 4$. In this way superspace is replaced by ordinary geometry in momentum-twistor space, but the amplitude will be thought of as a differential form.