Physics 253 B, Problem Set 1

September 17, 2019

(1) Plucker/Schouten/Cramer relations We will be doing some trivial linear algebra extensively in this course, so it's a good idea to have such manipulations under your belt! Consider two-dimensional vectors (they could be spinor-helicity variables) λ^α. Given any three vectors $\lambda_1^\alpha, \lambda_2^\alpha, \lambda_3^\alpha$, there must be a linear relation between them, e.g. we can expand λ_3 as a linear combination of λ_1, λ_2. Show that this relation is

$$\langle 23\rangle\lambda_1 - \langle 13\rangle\lambda_2 + \langle 12\rangle\lambda_3 = 0$$

Contract this equation with a fourth vector λ_4, to derive

$$\langle 13\rangle\langle 24\rangle = \langle 12\rangle\langle 34\rangle + \langle 23\rangle\langle 14\rangle$$

Generalize these relations to any $(k+1)$ vectors V^I in k dimensions where you should find

$$\langle V_2 \cdots V_k\rangle V_1 - \langle V_1 V_3 \cdots V_k\rangle V_2 + \langle V_1 V_2 V_4 \cdots V_k\rangle V_3 + \cdots = 0$$

and contracting with any other $(k-1)$ vectors gives us what are known as "Plucker relations". Show how you can also interpret this equation in terms of "Cramers rule", for inverting a matrix. Incidentally often for 2-vectors it is useful to remember the relation in the cyclically symmetric form $\langle 12\rangle\lambda_3 + \langle 31\rangle\lambda_2 + \langle 23\rangle\lambda_1 = 0$.

(2) Polarization vectors and spinor-helicity In class we contrasted the usual formalism of field theory using polarization vectors, to the spinor-helicity formalism. They are easily related as follows.Trading ϵ_μ for $\epsilon_{\alpha\dot\alpha}$, a polarization vector for a massless spin one particle of helicity -1 is

$$\epsilon_{\alpha\dot\alpha} = \frac{\tilde\mu_{\dot\alpha}\lambda_\alpha}{[\tilde\mu\tilde\lambda]}$$

where $\tilde\mu_{\dot\alpha}$ is some auxiliary spinor. Note that ϵ has helicity weight -1 as it should, and note that we have $\epsilon_\mu p^\mu = \epsilon_{\alpha\dot\alpha}\lambda^\alpha\tilde\lambda^{\dot\alpha} = 0$. The dependence on this auxiliary $\tilde{\mu}_{\dot{\alpha}}$ is disgusting but inevitable, this is how we see that there is no such thing as "the polarization vector" as a manifestly Lorentz-covariant object! But show that changing $\tilde{\mu}$ simply shifts $\epsilon_{\alpha\dot{\alpha}}$ by something proportional to $\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$, which is just the gauge redundancy we were forced to introduce. Instead of looking at the polarization vector itself, we can consider a gauge-invariant object like the field strength $f_{\mu\nu} = p_\mu \epsilon_\nu - p_\nu \epsilon_\mu$. Show that

$$f_{\alpha\beta\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} \lambda_\alpha \lambda_\beta$$

is manifestly independent of $\tilde{\mu}$. Repeat all these exercises for graviton polarization vectors.

(3) Fun with Parke-Taylor Factors Let's define

$$PT(1,2,\cdots,n) = \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

These objects satisfy interesting identities relating different orderings; we will discuss the physical signficance of these later, for now we're just practicing playing with bracket-ology. Show that

$$PT(1,2,3,4) + PT(1,3,2,4) + PT(1,3,4,2) = 0$$

More generally show that

$$PT(1,2,\cdots,n) + PT(1,3,2,\cdots,n) + \cdots + PT(1,3,4,\cdots,2) = 0$$

where we are just dragging the label "2" through to get an $(n-1)$ term expression. These are known as the "U(1) decoupling identities" (a clue as to the physical significance).

Incidentally there is a small useful point manipulating expressions, such as amplitudes, that have uniform little group weight for each label. It is sometimes convenient to write write $\lambda_a = t_a \times (1, z_a)$ (note that this picks a particular Lorentz frame). Show how this makes the Schouten relation a trivial algebraic identity. For our particular exercise $PT(1,2,\cdots,n) = (t_1 \cdots t_n)^{-2} \times 1/((z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1))$; this makes playing with the identities a little easier.

(4) Universal coupling of gravity In class we sketched the argument that the coupling of gravitons to any other massless particle X must be universal, that amplitudes for X having spin $\geq 5/2$ were inconsistent, and that multiple interacting spin 2 are impossible. Complete these arguments.

(5) Discovering $\mathcal{N} = 1, 2$ Supergravity In class, we showed that a single massless spin 3/2 + gravity is consistent, but that if we add a single scalar, the four-particle check fails unless we also add a massless fermion with couplings to the "gravitino" fixed to equal those of gravity. Extend this argument to the case where we add a massless spin 1, and discover the necessity of the "gauginos". Next, conisder the possibility of two masselss spin 3/2 particles, and see what is needed to make a consistnet theory, also coupled to other matter. You should discover in this way the consistent possible multiplets of $\mathcal{N} = 2$ supergravity.

(6) Six-point gluon scattering Use BCFW on the legs $6, 1$ to compute the $(1^+2^-3^+4^-5^+6^-)$ gluon amplitude. The three BCFW terms all turn out to be (unobviously) related simply by cyclic shift of the the labels, and you should find

$$\frac{[35]^4\langle 26\rangle^4}{\langle 61\rangle\langle 12\rangle[34][45](p_6+p_1+p_2)^2\langle 6|(p_1+p_2)|3][5|(p_6+p_1)|2\rangle}+\{i\to i{+}2\}+\{i\to i{+}4\}$$

Observe that the last two pole factors are "spurious poles", not corresponding to legal poles that vanish as $(p_i+p_{i+1}+\cdots p_{j-1})^2\to 0$. Try to explicitly confirm that the spurious poles cancel in the sum over the three terms. You may want to use Mathematica/Maple/Sage to play around with this. Repeat the computation for the $(1^+2^+3^+4^-5^-6^-)$ amplitude. Depending on how you do the recursion, you will get either a two-term or a three-term expression.

(7) Some Simple Super-BCFW Use BCFW to compute the 4-point super-amplitudes for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA. You should find

$$M^{SYM}=\frac{\delta^{2\times 4}\left(\sum_a\lambda_{a,\alpha}\tilde{\eta}_a^I\right)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle},\ M^{SUGRA}=\frac{\delta^{2\times 8}\left(\sum_a\lambda_{a,\alpha}\tilde{\eta}_a^I\right)}{stu}$$