Smooth Points on Positroid Varieties and Planar N=4 Supersymmetric Yang-Mills Theory

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Outline of Talk

-Grassmannians, Positroids, Positroid Varieties (pre-seminar)

-Pipe Dreams, Moves, Affine Pipe Dreams (mostly pre-seminar)

-Main Theorem on Smoothness (and Proof Outline)

-Main Theorem on Smoothness Characterization

-Deletion/Contraction and Atomicity

Grassmannians

Let V be an n-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$)

For any integer $0 \leq k \leq n$, $Gr_k(V)$ is the set of all n-dimensional linear subspaces (through the origin) of V.

$Gr_k(V)$ can be given the structure of a smooth manifold of dimension $k(n-k)$

In this talk, we will be concerned with $Gr_k(\mathbb{C}^n)$ denoted $Gr(k,n)$.

Example: (1) $Gr(0,n) = \{0\}$ the origin

Grassmannian Examples Exercises

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(3) $Gr(1,n) \cong$

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(3) $Gr(1, n) \cong \mathbb{P}^{n-1}$

(4) Thus by duality, $Gr(n-1, n) \cong \mathbb{P}^{n-1}$

Writing Grassmannians

We tend to write $Gr(k,n)$ as $GL(k)\backslash$[full rank $k \times n$ matrices], and think about it as k row vectors spanning a plane up to invertible basis transformation inside the plane.

Let $[n]$ denote the set $\{1, 2, ..., n\}$ and $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. Each full rank $k \times n$ matrix M has a point in $Gr(k,n)$ given by $GL(k)\backslash M$. Since M is full rank, it has a set of columns $\lambda \in \binom{[n]}{k}$ whose determinant is nonzero, so we can use $GL(k)$ to set λ to be the identity;

thus, $M \sim \begin{pmatrix} * & 1 & * & 0 & & 0 & * \\ * & 0 & * & 1 & & 0 & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & * & 0 & & 1 & * \end{pmatrix}$ where the columns that look like $(00\cdots 1 \cdots 0)^T$ are the ones in λ.

Grassmannian Cover

These $\binom{n}{k}$ choices of λ enumerate all the fixed points of the Grassmannian under the right torus action, that is, the action of $n \times n$ diagonal matrices by right multiplication.

Although an abuse of notation, in the following, we will use λ to denote the torus-fixed point on the Grassmannian corresponding to setting the columns λ to be the identity (and all other entries zero)

Furthermore, if we let the other columns of $*$'s be anything, so that these denote open sets U_λ, then these open sets are isomorphic to $\mathbb{C}^{k(n-k)}$. In fact, these open sets cover the Grassmannian.

Grassmannian Cover Exercise

Suppose we are in the Gr(2,4) Grassmannian, and let $\lambda = \{1, 3\}$.

Write down what the open affine patch U_λ looks like and what the point λ looks like.

Grassmannian Cover Exercise

Suppose we are in the Gr(2,4) Grassmannian, and let $\lambda = \{1, 3\}$.

Write down what the open affine patch U_λ looks like and what the point λ looks like.

Answer:

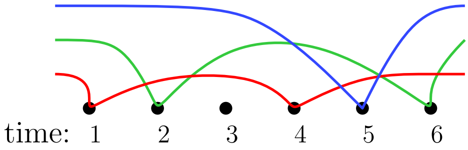
$$U_\lambda = GL_k \Big\backslash \left\{ \begin{pmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{pmatrix} \right\},$$

and

$$\lambda = GL_k \Big\backslash \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let's discuss positroids first. In particular, we describe bounded affine permutations and siteswaps, which are in bijection with positroids.

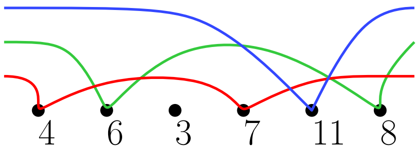
Here's is a juggling pattern:



This juggling pattern repeats every 6 seconds.
There are 3 balls, one for each color.

We can think of the dots as the hand; note that this is a simplified model in which there is only one hand.

This looks somewhat like a permutation. We could describe it as:



We have written below each point the moment in time when the ball thrown at that moment lands.

We write: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix}$. We call this a bounded affine permutation.

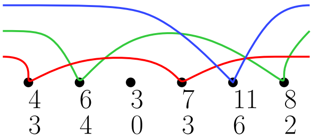
Definition (Bounded Affine Permutation)

Affine permutation: $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the periodicity condition $f(i+n) = f(i) + n$, $\forall i \in \mathbb{Z}$ where n is the period.
A *bounded* affine permutation also satisfies: $i \leq f(i) \leq i + n$.

Siteswaps

Since this juggling pattern is periodic, the choice of starting point was arbitrary.

We can create a permutation-invariant labeling by putting the units of time ahead of the present moment that the ball lands:



This second row of numbers is what we call the *siteswap*. We can obtain them by taking the bottom row and subtracting the top row: $f(i) - i$ in the bounded affine permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix} \mapsto \begin{pmatrix} 4 & 6 & 3 & 7 & 11 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 \\ 3 & 4 & 0 & 3 & 6 & 2 \end{pmatrix}.$$

Exercise

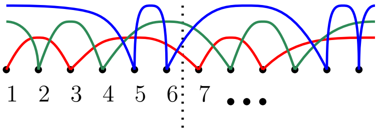
Problem: Consider the siteswap 242451. Write down its bounded affine permutation and draw a juggling pattern.

Exercise

Problem: Consider the siteswap 242451. Write down its bounded affine permutation and draw a juggling pattern.

Answer:

Bounded affine permutation $\begin{pmatrix} 2 & 4 & 2 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 8 & 10 & 7 \end{pmatrix}$



Siteswaps and Grassmannia

Now let's connect this to matrices and Grassmannians. Here's the connection: a juggling pattern of length n with k balls corresponds to a set of k by n matrices.

Definition (BAP Map)

We can define a map from full rank k-by-n matrices M to bounded affine permutations: we take each column of a matrix, and assign the earliest next column for which it's in the span of the following:

$$\tilde{g}_M(i) := \min\{j \geq i\colon \vec{c}_{i \bmod n} \in span(\vec{c}_{i+1 \bmod n}, \dots, \vec{c}_{j \bmod n})\}.$$

We only have to make a slight modification for siteswaps, in which case the map is:

$$\tilde{f}_M(i) := \min\{j \geq i\colon \vec{c}_{i \bmod n} \in span(\vec{c}_{i+1 \bmod n}, \dots, \vec{c}_{j \bmod n})\} - i$$

BAP Map Exercise

Suppose we have:

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

What is $\tilde{g}_M(i)$?

BAP Map Exercise

Suppose we have:

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

What is $\tilde{g}_M(i)$?

Answer:

$\tilde{g}_M(1) = 4$: e.g. the 1st column is not in the span of the 2nd and 3rd columns, but only in the span of columns 2-4.

Similarly, $\tilde{g}_M(2) = 6$, $\tilde{g}_M(3) = 3$, $\tilde{g}_M(4) = 7$, $\tilde{g}_M(5) = 11$, $\tilde{g}_M(6) = 8$. We recognize this as the bounded affine permutation from before.

Definition of Positroid Variety

Positroid varieties, denoted Π_f are subsets of Grassmannia defined by the cyclic rank conditions imposed by siteswaps, or equivalently, bounded affine permutations f (f might also denote the siteswaps, but should be clear from context). More precisely:

Definition (Positroid Variety)

For a given siteswap f, we can define the positroid variety Π_f using the map $\tilde{f}_M(i)$ from matrices to siteswaps we gave above:

$$\Pi_f = \{M \in Gr(k, n) \mid \tilde{f}_M(i) \leq f(i)\}$$

(The open positroid variety is $\Pi_f^\circ = \{M \in Gr(k, n) \mid \tilde{f}_M(i) = f(i)\}$)

We can give a second definition of positroid varieties using the ranks of all cyclic columns. In particular this will require us to show how to get *all* cyclic ranks $rank[i,j]$, not just $rank[i,f(i)]$, from the siteswaps.

First, let's switch back to doing bounded affine permutations. Our bounded affine permutation was $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix}$.

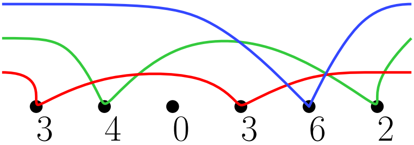
Let's put our bounded affine permutation into a matrix, and insert the first number:
$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$.

Let's do the second column as well:
$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.
Note that like a normal permutation matrix there is a single one in each row and column.

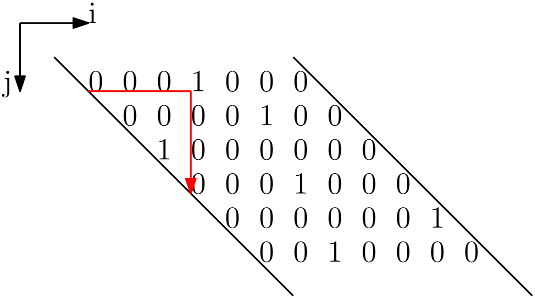
This is often times drawn with lines running from northwest to southeast showing the allowed region with horizontal axis i increasing to the right and vertical axis j increasing to the south:

$$
\begin{array}{l}
\rightarrow i \\
\downarrow j
\end{array}
\qquad
\begin{array}{cccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & & \\
 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & & & \\
 & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
 & & & & & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}
$$

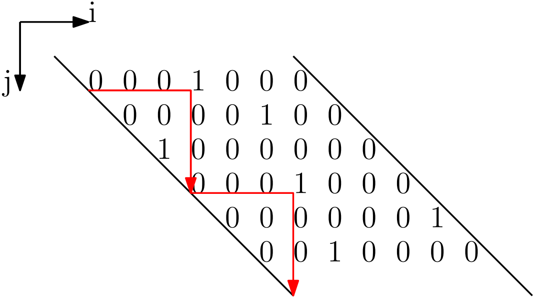
Let's recall our juggling pattern:



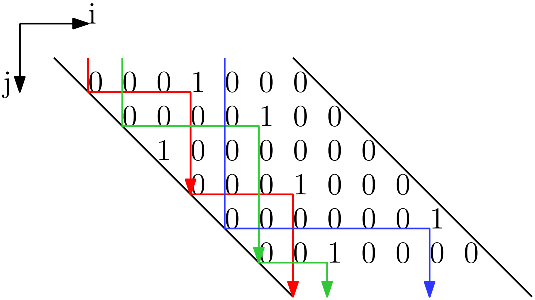
Let's draw the line on the permutation matrix corresponding to the first throw of the ball at time $t = 1$ that lands at time $t = 4$:



Let's draw the entire path of the ball in the juggling pattern represented by the red line:

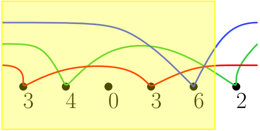


Let's show what the lines look like when we include all the balls in the juggling pattern:

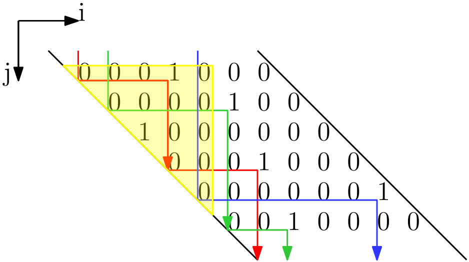


It's clear that what's going on is that we draw the lines horizontally from the left edge of the diagram until the line hits a 1, and then we go south until it hits the left edge of the diagram again, at which point we go horizontally right again.

So finally, let's describe how we calculate the cyclic ranks. Let's do an example. Let's take the juggling pattern and let's look at $[i, j] = [1, 5]$:

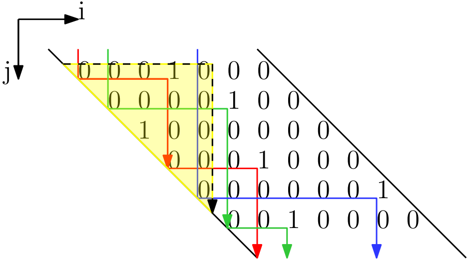


corresponding to the highlight in the permutation matrix:

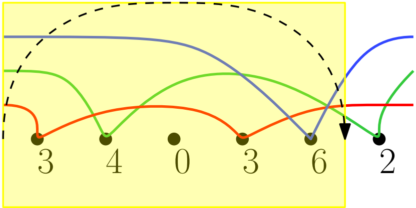


There are two 1's in the highlighted portion: at (1,4) and (3,3). These are precisely the places of balls that are both thrown and landed within this interval $[1, 5]$.

Let's be clearer. Let the dotted black line in the permutation diagram be a hypothetical ball that would be thrown at time $t = 1$ and land at time $t = 5$:



This would correspond to a hypothetical ball of something like the dotted line below:



We have drawn it to start a little before time $t = 1$ and land a little after $t = 5$ to emphasize that it contains this interval (it includes both $t = 1$ and $t = 5$).

Definition of Positroid Variety with All Cyclic Ranks

Here's the formula for rank:

$rank[i,j] = |[i,j]| - \{\#1's \text{ to SW}\} = (j-i+1) - \{\#1's \text{ to SW}\}$. "SW" means "southwest."

Now, we can finally give the second definition of positroid variety:

Definition (Positroid Variety)

$\Pi_f = GL_k \backslash \{M \subseteq M_{k,n} | rank([i,j]) \leq |[i,j]| - \#1's$ southwest of (i,j), $i \leq j \leq i+n\}$ where $rank([i,j])$ is defined cyclically

This should make sense from what we have just explained. In our example, $rank[1,5] = |[1,5]| - 2 = [(5-1)+1] - 2 = 3$. So the rank of this interval is 3.

T-duality

It is easy to depict T-duality using these diagrams. First, here's the definition of T-duality given in [Hu-Yeats23]:

Definition

The T-duality map from loopless decorated permutations on $[n]$ to co-loopless decorated permutations on $[n]$ is defined as

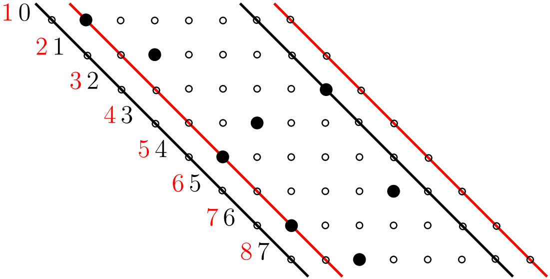
$$\pi \mapsto \hat{\pi}$$

$$(a_1, a_2, \ldots, a_n) \mapsto (a_n, a_1, \ldots, a_{n-1})$$

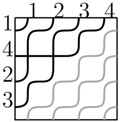
where the permutations are written in one-line notation, and any fixed points in $\hat{\pi}$ are declared to be loops. That is, for a given loopless π, we have $\hat{\pi}(i) = \pi(i-1)$ where all fixed points are loops, and we call $\hat{\pi}$ the T-dual decorated permutation.

Let's use the example of a bounded affine permutation $3, 8, 6, 5, 10, 7$. Writing the BAP and siteswap below it, T-duality sends:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 8 & 6 & 5 & 10 & 7 \\ 2 & 6 & 3 & 1 & 5 & 1 \end{pmatrix} \mapsto {\color{red}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 8 & 6 & 5 & 10 \\ 0 & 1 & 5 & 2 & 0 & 4 \end{pmatrix}}$$



Pipe Dreams



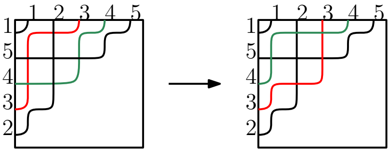
Definition (Pipe Dream in S_n)

A pipe dream in S_n is a diagram in an n-by-n square where each box can be filled by one of two types of tiles, elbows and crosses, such that all crosses occur above the southwest-northeast diagonal.

Thus, we can think of the grid as a set of pipes that begin on the north and east edges and end on the west and south edges (with the south to east tiles being all elbows)

We called the pipe dream *reduced* if none of the pipes cross twice. We will focus only on the reduced case in the following.

Moves on Pipe Dreams



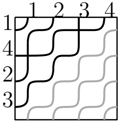
Definition

A **cross-elbow move** *(in the following: "***move***," for short) in a pipe dream is an interchange of a cross tile and an elbow tile that preserves the connectivity of the pipes - in other words, that leaves the permutation invariant. We think of the cross as "moving" to the position where the elbow it is replacing was located.*

If the pipe dream has no possible moves, so is the unique pipe dream for a particular permutation, then we call the pipe dream **rigid**.

Moves on Pipe Dreams Exercise

Consider the following pipe dream:



Apply all possible moves to obtain the set of all possible pipe dreams for this permutation (1423).

Moves on Pipe Dreams Exercise

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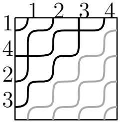
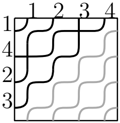
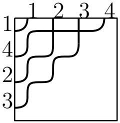
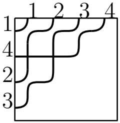


Apply all possible moves to obtain the set of all possible pipe dreams for this permutation (1423).

Answer:

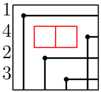




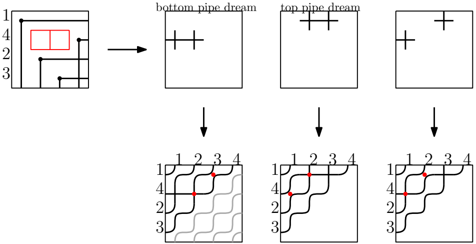


Rothe Diagrams

This is not central to our story here, so we just mention in passing, that we can draw a *Rothe diagram* associated with π_{1423}, given by drawing lines south and east from each 1 in the permutation matrix:

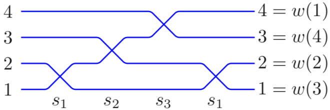


If we turn the boxes into crosses and push them to the left, we get the "bottom pipe dream"; to the top is the "top pipe dream." (We can then apply Bergeron-Billey moves to get all pipe dreams)



Wiring Diagrams

Another way to think about pipe dreams is as wiring diagrams. Here is an example from Postnikov:



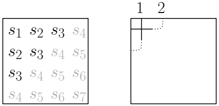
With transpositions $(i, i+1)$ denoted by s_i, we see that the wiring diagram gives us a word $s_1 s_3 s_2 s_1$ which produces this permutation (we read the word in the order such that it acts on the right):

$$s_1 s_3 s_2 s_1(1234) = s_1 s_3 s_2(2134) = s_1 s_3(2314) = s_1(2341) = 3241.$$

In this context, reduced means that no two lines cross twice. For these words, reduced corresponds to it being the shortest word producing a given permutation.

Pipe Dreams - Wiring Diagrams

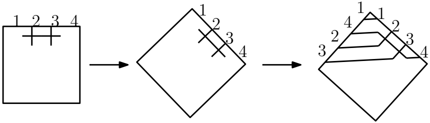
Now let's see what the transpositions are in the square from before:



We read from top to bottom and right to left: $s_3 s_2 s_3 s_1 s_2 s_3$.

The second figure shows that if we just had the northwesternmost cross, it would switch pipes 1 and 2, so it makes sense for it to be s_1.

For the example above of 1423, all pipe dreams give the same word $s_2 s_3$. We can thus view the pipe dream as a rotated wiring diagram producing a reduced word:



We verify: $s_2 s_3(1234) = s_2(1243) = 1423$.

Words

Definition (see Knutson-Miller 03)

Let Σ denote the set of transpositions $s_1, \ldots, s_n$ where s_i switches i and $i+1$. A word of size m is an ordered sequence $Q = (\sigma_1, \ldots, \sigma_m)$ of elements of Σ. An ordered subsequence P of Q is called a **subword** *of Q.*

P **represents** *$\pi \in \Pi$ if the ordered product of the simple reflections in P is a reduced decomposition for π.*

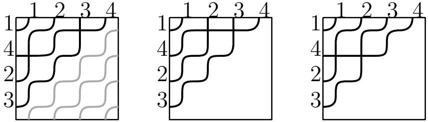
P **contains** *$\pi \in \Pi$ if some subsequence of P represents π.*

The subword complex $\Delta(Q, \pi)$ is the set of subwords $Q \backslash P$ whose complements P contain π.

Square pipe dreams correspond bijectively to reduced words and embeddings into square words.

Wiring Diagrams/Words Exercise

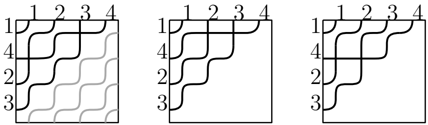
Recall the following figure from our last exercise:



What are all the subwords of the big word $s_3 s_2 s_3 s_1 s_2 s_3$ of each pipe dream? (let's call them (1), (2) and (3)):

Wiring Diagrams/Words Exercise

Recall the following figure from our last exercise:



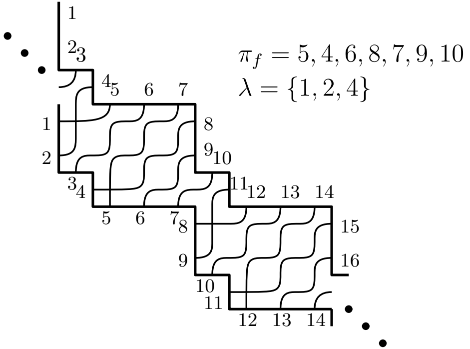
What are all the subwords of the big word $s_3 s_2 s_3 s_1 s_2 s_3$ of each pipe dream? (let's call them (1), (2) and (3)):

Answer:

(1) $s_3 s_2 s_3 s_1 s_2 s_3$: $s_2 s_3$
(2) $s_3 s_2 s_3 s_1 s_2 s_3$: $s_2 s_3$
(3) $s_3 s_2 s_3 s_1 s_2 s_3$: $s_2 s_3$
So the answer is that they are all $s_2 s_3$.

Affine Pipe Dreams

We have an analogue of pipe dreams for bounded affine permutations called *affine pipe dreams*. Here is an example:

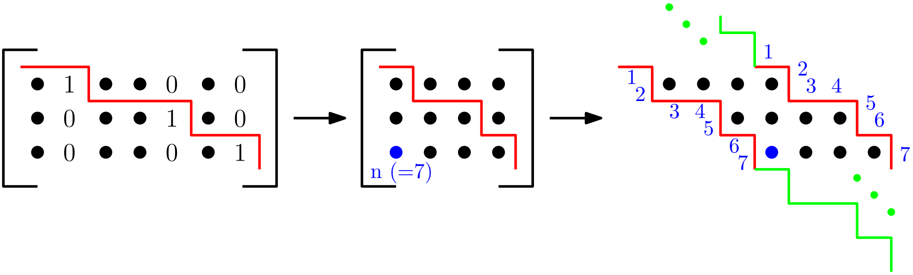


We show how this works via an example of a siteswap 2225377 (bounded affine permutation $34598, 13, 14$) inside Gr(3,7) on U_λ with $\lambda = \{2, 5, 7\}$.

Affine Pipe Dreams

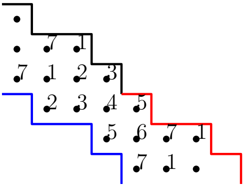
We first write the U_λ corresponding to the point $\lambda = \{2, 5, 7\}$. The red line is called the distinguished path - you move horizontally until you hit an identity column, at which you move down.

Next, we (1) redraw this line shifted up and right, creating a finite strip (2) label the bottom line with the column number, and the top line with the corresponding number.

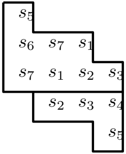


Affine Pipe Dreams

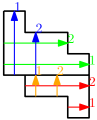
We thus see what the transpositions should be. In analogy with the usual case, they run diagonally southwest to northeast, with s_1 being where we can switch pipes 1 and 2:



If we just look at one block, denoting the transpositions by s_i again, it is:



To get the bottom pipe dream, we have to pick a way of reading this. There are equivalent ways of reading it, given by:



In other words, if we choose the direction of red followed by green, then the ordering will be $s_5, s_2s_3s_4, s_7s_1s_2s_3, s_6s_7s_1, s_5$.

If we apply this word to $(k+1, k+2, \dots k+n)$, then we get the bounded affine permutation correspond to $+n$ for the λ columns $2, 5, 7$ and $+0$ for all other columns.

This shows that this pipe dream shape represents the neighborhood U_λ.

Here's how reading it this way produces a word for siteswap 2225377 (bounded affine permutation $34598, 13, 14$):

We start with:

1	2	3	4	5	6	7	8(=1)
3	4	5	9	8	13	14	

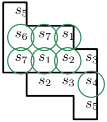
Since the big word is $s_5, s_2s_3s_4, s_7s_1s_2s_3, s_6s_7s_1, s_5$, we find the first one where there was a transposition with $f(i) > f(j), i < j$. This first one is s_4 which switches the 98 to 89:

1	2	3	4	5	6	7	8(=1)
3	4	5	9	8	13	14	
3	4	5	8	9	13	14	10

We continue on until we get the most general permutation:

1	2	3	4	5	6	7	8(=1)
3	4	5	9	8	13	14	
3	4	5	8	9	13	14	10
7	4	5	8	9	13	10	14
4	7	5	8	9	13	10	11
4	5	7	8	9	13	10	11
4	5	7	8	9	10	13	11
6	5	7	8	9	10	11	13
5	6	7	8	9	10	11	12

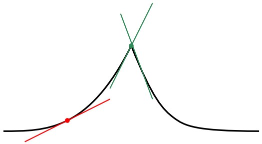
Keeping track of which transpositions we used, we get:



If we put crosses where the circles are, and elbows elsewhere, then we get the bottom pipe dream.

Smooth vs. Singular

We are interested in determining whether a positroid variety is singular or smooth. Intuitively, it is the notion that the tangent space is not "too big":



In this picture, the manifold drawn in black is 1-dimensional, and the red point is a smooth point since the tangent space is the right dimension, but the tangent space at the green point is spanned by the two lines, so it is too big.

Smooth Definitions

For completeness, we give the following definition of smoothness:
When the maximal ideal has number of generators equal to the dimension of the ring, that is, $dim I(a) = dim O_{X,a}$ So for a variety X, a part $a \in X$ is smooth if its ring of local function $A(X)_{I(a)}$ is regular, that is, if $dim T_a X = dim Hom_K(T_a X, K) = dim I(a)/I(a)^2 = codim_X\{a\} = dim X = dim A(X)_{I(a)}$. Equivalently, X is smooth at a if its tangent cone is the same as its tangent cone at a.
We often test smoothness using some sort of Jacobi condition, e.g.:

Theorem

Affine Jacobi Criterion
Let $a \in X$ be a point on an affine variety $X \subseteq \mathbb{A}^n$ and let $I(x) = \langle f_1, ..., f_n \rangle$. Then
X is smooth at $a \Leftrightarrow$ the rank of the $r \times n$ Jacobian matrix $(\frac{\partial f_i}{\partial x_j}(a))_{ij}$ is at least $n - codim_X\{a\}$ (so it's equal to $n - codim_X\{a\}$

Singular at Points

Theorem

We can check whether a positroid variety Π_f is singular (or smooth) just by testing singularity at the $\binom{n}{k}$ fixed points given by the span of k coordinate spaces.

Denoting the fixed points of the torus action by superscript T, we have:
$(\Pi_f)_{sing} \neq \emptyset \leftrightarrow (\Pi_f)^T_{sing} \neq \emptyset$

So we have reduced checking singularity (or smoothness) to a finite set of points $\{\lambda\} = \binom{[n]}{k}$!

In what follows, we will check these by checking on the affine open cover of the U_λ described earlier (referenced in a moment at affine pipe dreams. Clearly, this will include all fixed points $\{\lambda\}$.

Statement of Main Theorem on Smoothness

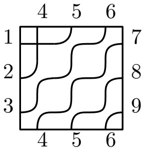
Here is the main theorem on smoothness for a positroid variety at a torus-fixed point.

Theorem (Main Theorem on Smoothness)

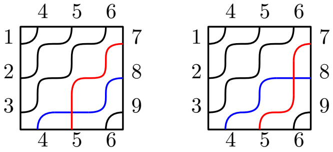
A positroid variety Π_f is smooth at the point λ (it meets λ and is smooth there) if and only if there is a single affine pipe dream representative of the affine permutation f on the open set U_λ. Specifically Π_f meets the point λ if and only if there exists (at least) one affine pipe dream, and it is smooth if and only if that affine pipe dream is unique. Thus, we can take Π_f and look at pipe dreams for all choices of λ to test for smoothness.

Main Theorem on Smoothness Example

Here's a basic example to make the statement of the Main Theorem on Smoothness easy to understand:



Smooth



Singular

On the left we have $\lambda = \{1, 2, 3\}$ and $g = 5, 4, 6, 7, 8, 9$. This is a smooth point on positroid variety Π_g.

On the left we have $\lambda = \{1, 2, 3\}$ and $g = 4, 5, 6, 8, 7, 9$. This is a singular point on positroid variety Π_g.

Proof of Main Theorem on Smoothness

$$\begin{aligned}[\Pi_f \cap U_\lambda \subseteq U_\lambda] &\overset{(1)}{=} [X_{v(f)} \cap X_\circ^{v(\lambda)} \subseteq X_\circ^{v(\lambda)}] \\ &\overset{(2)}{=} [X_{v(f)}]|_{v(\lambda)} \\ &\overset{(3)}{=} \sum_R \prod_{r \in R} \hat{\beta}_r \in H_T^*(U_\lambda) \cong \mathbb{Z}[y_1, \ldots, y_n]\end{aligned}$$

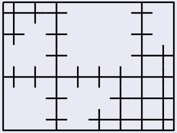
Main Theorem Preliminaries - NW/SE Partitions

Definition

A pipe dream in a rectangular shape **reduces to NW/SE partitions** *if, after deleting all entire rows and columns of crosses, the result is a partition in the southeast and northwest corners.*

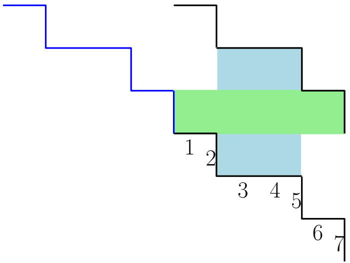
Example

The following is an example of a rectangular pipe dream that reduces to NW/SE partitions:



Main Theorem Preliminaries - Maximal Rectangles

Since the following parts involve the notion of a maximal rectangle, we include a figure to display what the maximal rectangles look like in our pipe dream shapes:



Specifically, the southwest corner of the rectangle is a corner of the lower boundary of the shape, and the northeast corner of the rectangle is a corner of the upper boundary of the shape.

Main Theorem

Theorem (Main Theorem)

Let $\delta \in PD_{(\Pi_f,\lambda)}$ be an affine pipe dream for $\Pi_f \cap U_\lambda$. If each maximal rectangle in δ reduces to NW/SE partitions, then Π_f is smooth at λ; otherwise, it is singular.

Proof.

This theorem consists of showing two parts:
(1) Showing that smoothness can be determined just by looking at the maximal rectangles in an affine pipe dream. This is proven in Proposition 1 below.
(2) Showing that the thing that needs to be looked at on each maximal rectangle is that each maximal rectangle is a NW/SE partition. This is proven in Theorem 3 below.
However, first we give the full proof, referencing both Proposition 1 and Theorem 3 as needed. □

Main Theorem Proof

Proof.

We want to show that smoothness of (Π_f, λ) is equivalent to δ having no moves *in any maximal rectangle*. But we already have Theorem 1, which says that smoothness of (Π_f, λ) is equivalent to its pipe dream having no moves (anywhere), and, by Proposition 1, all moves exist within maximal rectangles. Thus, if all maximal rectangles have no moves, then the pipe dream for (Π_f, λ) has no moves. The other direction, that the pipe dream for (Π_f, λ) having no moves implies that all its maximal rectangles have no moves, is automatic.

Therefore, smoothness of (Π_f, λ) is equivalent to δ having no moves in any maximal rectangle. Thus, we just have to show that rigid rectangles reduce to NW/SE partitions and the proof will be complete, and this is shown in Theorem 3. $\square$

Proposition 1

Proposition

Every cross-elbow move exists within some maximal rectangle. More precisely, every pair consisting of a cross and a near-miss where this cross can move is contained within a maximal rectangle.

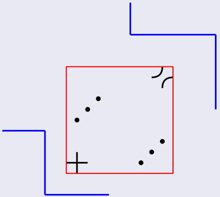
Trivially, this implies that the portions of the two pipes involved in the move, starting from the cross and running all the way to the near-miss, are entirely contained in this maximal rectangle.

In fact, the proof below shows the stronger statement that if we consider a cross and the set of all places where it can move, these will all be contained in a maximal rectangle.

Proposition 1 Proof

Proof.

The very notion itself of nonrigidity is that there is a cross to the southwest and a near-miss of the pipes in that cross somewhere to the northeast (or vice-versa with southwest and northeast switched). In the sorts of shapes given by affine pipe dreams we are considering, such a cross and then near-miss combination must exist within a full rectangle because all the jagged (or convex) parts of the shape are in the northeast and southwest. This is depicted in the red rectangle in the following figure:



□

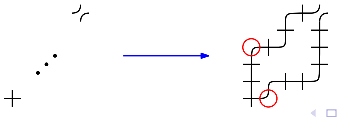
Theorem 3 Preliminary

Proposition

A rectangular pipe dream is rigid if and only if: (1) no cross has both an elbow to its north as well as to its east, and (2) no cross has both an elbow to its south as well as to its west.

This is essentially a "double-Le condition," and the proof is similar to statements in Section 19 of Postnikov or Section 4.3.3 of Snider. As hinted in Snider, one of these conditions basically corresponds to the bottom pipe dream (see Appendix B), so the other would correspond to the top pipe dream.

The idea of the proof is captured by the following picture, which shows that if a cross has a move to a northeast near-miss, then this cross must bend to its north and east.

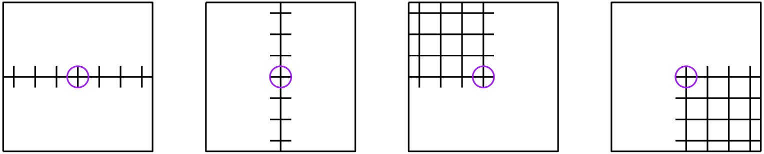


Theorem 3

Theorem

All rigid rectangles reduce to NW/SE partitions.

What we will prove below is that every cross (schematically shown by circling the crosses in red) is contained in one of the four configurations below.



Deletion and Contraction

Why consider deletion/contraction?

"Deletion and contraction of an element are fundamental operations in many areas...there is a tremendous power in proofs "by deletion and contraction" which proceed by induction on the number of elements, and by putting a structure together from the information given by deletion and contraction of the same element." (Lectures on Polytopes, Gunter Ziegler, 1995)

Deletion and Contraction on Grassmannians

The Grassmannian is represented by full-rank k-by-n matrices (up to $GL(k)$ action on the left).

Then there is a circle action on the right via matrix multiplication by

$$S_i = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & z & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

such that this scales i-th columns of $\{M\}$ by z.

Deletion and Contraction on Grassmannians

There is a "Pascal recurrence" on Grassmannians:

Proposition

Let S_i act on $\{M\} = Gr(k,n)$ on the right. Let

$$\Pi_{contr,i} := rowspan \left\{ \begin{pmatrix} & & & 0 & & & \\ & * & & 0 & & * & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \right\} \cong Gr(k-1, n-1) \text{ and}$$

$$\Pi_{del,i} := rowspan \left\{ \begin{pmatrix} & 0 & \\ * & 0 & * \\ & 0 & \end{pmatrix} \right\} \cong Gr(k, n-1)$$

where the rest of the matrix is full-rank.
Then the fixed point set under the S_i action is:

$$Gr(k,n)^{S_i} = \Pi_{contr,i} \sqcup \Pi_{del,i} \cong Gr(k-1,n-1) \sqcup Gr(k,n-1).$$

Deletion of Positroid Varieties

Let's name as projectionless contraction the intersection:
$Contr_i(\Pi_f) = \Pi_f \cap \Pi_{contr,i}$, and similarly the projectionless deletion as $Del_i(\Pi_f) = \Pi_f \cap \Pi_{del,i}$.
There is another description of deletion/contraction of positroid varieties:

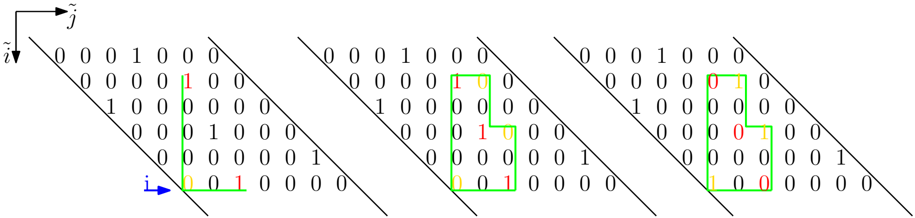
Proposition

The projectionless deletion $Del_i(\Pi_f)$ of the i-th column of a positroid variety Π_f is equal to $\Pi_{f'}^{d(i)} :=$ the largest positroid variety $\Pi_{f'}$ contained in Π_f such that $f'(i) = 0$. (By largest, we mean that $\Pi_{f'}$ is not properly contained inside another positroid variety $\Pi_{f''}$ properly contained in Π_f.)

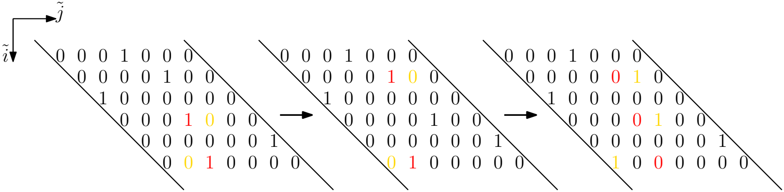
The projectionless contraction $Contr_i(\Pi_f)$ of the i-th column of a positroid variety Π_f is equal to $\Pi_{f'}^{c(i)} :=$ the largest positroid variety $\Pi_{f'}$ contained in Π_f such that $f'(i) = n$.

Deletion of Positroid/Varieties Siteswaps

Let's see what deletion/contraction do to the siteswap via the diagonal diagrams. Here is an example of deletion of the ith entry:



Or, one by one:



Deletion of Siteswaps Algorithm

The algorithm for the picture above is as follows:

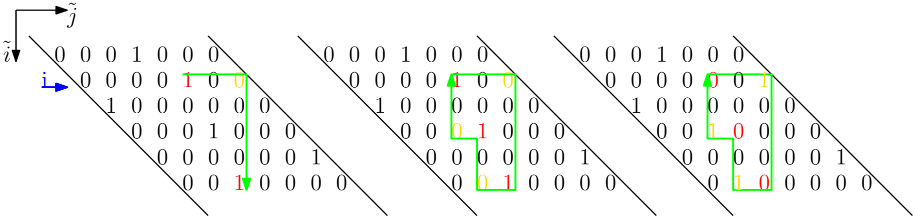
Let g denote the bounded affine permutation.

Start with i and $g(i)$, we want $g(i) = i$ since this means $f(i) = 0$.

Test $i-1$, $i-2$,... successively until the condition is reached. So we begin by testing $i-1$: if $i <= g(i-1) < g(i)$, then transpose i and $i-1$: define the new $g(i) := g(i-1)$ and $g(i-1) := g(i)$. So now $g(i)$ is a little closer to i. Continue until $g(i) = i$.

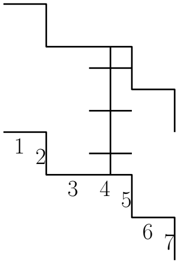
Contraction of Siteswaps Algorithm

Here we present an example of contraction of the *i*th entry, which is the same but reflected (via Grassmannian duality):



Deletion of Positroid Varieties on Affine Patches

Here's what $\Pi_{del,i}$ for $i = 4$ looks like for the point $\{2, 5, 7\}$ in Gr(3,7):



Deletion of a Positroid Variety at a Point

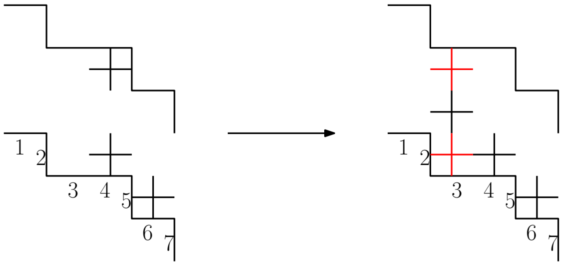
We can ask the question of what the deletion of a positroid variety Π_f at a given column i looks like on a patch U_λ. We already saw that this was given by the intersection: $del_i(\Pi_f) = \Pi_f \cap \Pi_{del,i}$, where $\Pi_{del,i}$ is the pipe dream with elbows everything except crosses along the column i.

So the question is: what does this intersection look like on U_λ?
Answer: it is given by filling in the pipe dream for Π_f on U_λ with crosses along the ith columns.

This prompts the immediate question: if there are multiple such pipe dreams for Π_f on U_λ, which one do we use?
Answer: we use those pipe dreams with the maximal number of crosses along column i.

Deletion of a Positroid Variety at a Point

For example if we want to delete column 3:



That is, we move as many crosses to that column (3) as possible, and fill in the rest of the squares in that column with crosses (denoted by the red cross here). This can be thought of as intersecting with $\Pi_{del,3}$.

The Ordering

Deletion and contraction give us an ordering on positroid varieties: we can consider a covering relation where $\Pi_{f_2} \lessdot \Pi_{f_1}$ if Π_{f_2} is obained from Π_{f_1} by deletion or contraction of one of the coordinates i.

However, we will be dealing in the following with an ordering on pairs (Π_f, λ), which we think of as Π_f restricted to the open patch U_λ.

In this ordering, $(\Pi_{f_2}, \lambda_2) \lessdot (\Pi_{f_1}, \lambda_1)$ if deletion or contraction of one of the coordinates i takes Π_{f_1} to Π_{f_2} and takes the point λ_1 to λ_2.

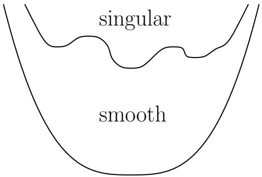
Smoothness in the Del/Contr Ordering

Smooth positroid varieties remain smooth under deletion and contraction:

Proposition

If λ is smooth on Π_f, then $del_i(\lambda)$ is smooth on $\Pi_{del_i(f)}$ and $contr_i(\lambda)$ is smooth on $\Pi_{contr_i(f)}$. Or this can be stated contrapositively: if $del_i(\lambda)$ is singular on $\Pi_{del_i(f)}$ or $contr_i(\lambda)$ is singular on $\Pi_{contr_i(f)}$, then λ is singular on Π_f.

This means that in the ordering (given by deletion and contraction) of positroid varieties and points, the smooth versus singular pairs can be divided in the following way:



Smoothness in the Del/Contr Ordering Proof 1

Geometric proof.

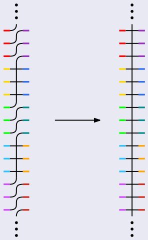
By e.g. Billey Braden, if the group G is reductive, then whenever G acts on a regular set X, the fixed point set X^G is also regular. In our case, $G = T$ is the torus, so the conclusion follows.

In particular, if Π_f is smooth at p, then $\Pi_f^{\mathbb{C}^\times}$ is smooth at p, and $\Pi_f^{\mathbb{C}^\times} = (\Pi_f \cap \Pi_{del,i}) \cup (\Pi_f \cap \Pi_{contr,i}) = Del_i(\Pi_f) \cup Contr_i(\Pi_f)$. □

Smoothness in the Del/Contr Ordering Proof 2

Combinatorial proof.

We can also give a combinatorial proof of this proposition using the machinery of affine pipe dreams. The proof goes by breaking the ith column or row into alternating sets of crosses and elbows:



Analyzing in this way, we find that filling in the column with crosses in this way cannot produce a move that did not previously exist.

□

Smoothness in the Del/Contr Ordering Proof 3

Proof using the Main Theorem.

Recall the definition of "reducing to NW/SE partitions": this means having a rectangular shape where the northwest and southeast were partitions, and additionally, any number of rows or columns could be filled in with crosses; everything else is elbows. The Main Theorem says that (in this smooth case) all maximal rectangles reduce to NW/SE partitions; now, taking any one of these maximal rectangles (which reduce to NW/SE partitions) and filling in a row or column with crosses is still precisely a shape that reduces to NW/SE partitions. Therefore, it is still smooth. □

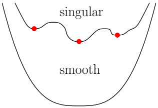
Atomicity

Next, we describe the singular pairs that are lowest in the order:

Definition (Atomic Positroid Pair)

A positroid variety Π_f is **atomic** *on an open patch U_λ if it is singular at λ, but any deletion or contraction (of any column) causes this positroid variety to be smooth (or empty) at λ. We call (Π_f, λ) an* **atomic positroid pair**.

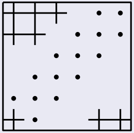
By definition, any deletion or contraction of an atomic positroid pair results in a smooth positroid pair (see the figure below). The order we defined on positroid pairs has the property that for any given pair, as we go down in the order, we can only delete or contract a finite number of times. Because of this, any singular pair lies over an atomic pair, i.e. these minimal elements generate the order ideal.



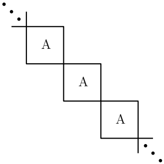
Atomicity

Proposition (Characterization of Atomic Pairs)

The atomic positroid pairs have affine pipe dreams that look like the following:



More precisely, the affine pipe dreams looks like the following, with each square labeled "A" being the same, and taking the form of the previous figure:

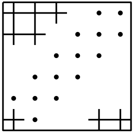


Atomicity

Specifically:
(1) The atomic positroid pairs have pipe dreams in a square shape: so they are in $Gr(k, 2k)$ and on points where λ consists of k consecutive columns.

(2) In this pipe dream, there is a single cross along the southwest to northeast (longest) diagonal, with the diagonals directly adjacent to this longest diagonal being free of any crosses.

(3) Besides this one cross, there are partitions of crosses in the northwest and southeast corners, and no others.



Atomicity Proof

Proof.

The proof goes by showing that in order for any deletion or contraction to eliminate all moves, the crosses must be able to reach any column or row - this is only possible along the main diagonal. Everything else must be rigid (no moves), so they must be partitions in the northwest and southeast.

Note: any singular positroid variety at a point can be deleted/contracted until it reaches the atomic configuration, so being able to reach this atomic configuration under deletion and contraction is equivalent to being singular at that point. However, it is also easy to show this directly, by showing that any pipe dream with moves can be brought to this form. □

Scattering Amplitude

What is an "Amplitude"?
Essentially, "a quantity that tells you about the probability of a particle physics experiment happening."

More technically: M is called the "scattering amplitude". It is calculated via the S-matrix.
The S-matrix is defined as a matrix element for a process where initial states i (almost always 2 particle colliding) turn into states f, where i are assumed to be free at $t = -\infty$ and f at $t = \infty$.

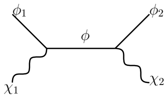
We can write it as:
$\langle f|S|i\rangle_{Heisenberg} = \langle f; t = \infty|i; t = -\infty\rangle_{Schrodinger} =$
$lim_{t_1 \to -\infty, t_2 \to \infty} \langle f|e^{iH(t_2 - t_1)}|i\rangle$.
We can calculate S-matrices using the theoretical tools of Feynman diagrams.
Actually, the scattering amplitude is really in the T-matrix:

$S = 1 + iT$, so we separate $\langle f|i\rangle$, a process that does nothing. The relation for the scattering amplitude is:
$\langle f|T|i\rangle = (2\pi)^2\delta(\sum p)M$. The delta function imposes momentum conservation of the process.

Quantum Field Theory

In order to do calculations in quantum field theory, you need to specify which "theory" = which particles are colliding and what interactions are possible. This is given in standard QFT by specifying a "Lagrangian (density)." Example: $\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{1}{2}\chi(\Box + m^2)\chi + \frac{g}{3!}\phi^2\chi$.

The essential gist: the Lagrangian tells you to draw a bunch of diagrams, like:



And the Feynman rules tell you how to take each diagram and write down an algebraic expression.

In this example, one would write down something like $\frac{g^2}{p_\phi^2 - m_\phi^2}$.

N=4 SYM

The theory we will be working with today is the "most symmetric QFT possible": N=4 Supersymmetric Yang-Mills.
Supersymmetry means that there is a symmetry of the Lagrangian of exchange of fermions (even-dimensional representation, half-integer spin/helicity) and bosons. So there are supersymmetry generators (Lie algebra generators of the Lie group).
N=4 SYM is a theory with 16 particles:

$$\underbrace{a}_{1\,\text{gluon}\,g^+},\quad \underbrace{a^A}_{4\,\text{gluinos}\,\lambda^A},\quad \underbrace{a^{AB}}_{6\,\text{scalars}\,S^{AB}},\quad \underbrace{a^{ABC}}_{4\,\text{gluinos}\,\lambda^{ABC}\sim\bar{\lambda}_D},\quad \underbrace{a^{1234}}_{1\,\text{gluon}\,g^-}$$

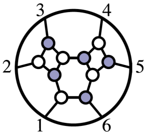
It is unique because it is maximally supersymmetry: every field can be moved to another field via applications of generators. Thus, we can write the entire thing as a superfield:

$$\Omega = g^+ + \eta_A\lambda^A - \frac{1}{2!}\eta_A\eta_B S^{AB} - \frac{1}{3!}\eta_A\eta_B\eta_C\lambda^{ABC} + \eta_1\eta_2\eta_3\eta_4\, g^-$$

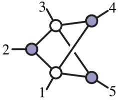
And we have a notion of a superamplitude (curly A) with the superfield as inputs, from which we can project out the amplitudes:

$$A_n(1^+\dots i^-\dots j^-\dots n^+) \;=\; \left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{iA}}\right)\left(\prod_{B=1}^{4}\frac{\partial}{\partial\eta_{jB}}\right)\mathcal{A}_n(\Omega_1,\dots,\Omega_n)\Big|_{\eta_{kC}=0}$$

We will be dealing with *planar* N=4 SYM, where the meaning of planar is exactly as it sounds. The diagrams look like:



and not:



In planar N=4, SYM, there was a formula known as the "Link Formula" that allows you to calculate the amplitude using positroid varieties:

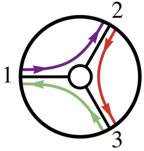
$$f_\sigma^{(k)} = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_d}{\alpha_d}\, \delta^{k\times 4}(C\cdot\widetilde{\eta})\delta^{k\times 2}(C\cdot\widetilde{\lambda})\delta^{2\times(n-k)}(\lambda\cdot C^\perp)$$

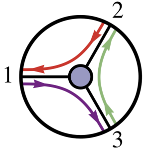
The C-matrix parameterizes a positroid variety - this is where positroids enter the story.

Note that we care about the case when the delta functions exactly equal the number of integrations: these are called "on-shell functions."

On-Shell Diagrams and Positroid Varieties

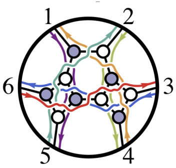
This is the easiest way to connect these planar N=4 SYM scattering diagrams with positroid varieties.
This is known as the left-right paths. Every time you hit a white vertex, turn left. Every time you hit a black vertex, turn right:



$\Leftrightarrow \begin{pmatrix}1 & 2 & 3\\ \downarrow & \downarrow & \downarrow\\ 2 & 3 & 1\end{pmatrix}$ and 

$\Leftrightarrow \begin{pmatrix}1 & 2 & 3\\ \downarrow & \downarrow & \downarrow\\ 3 & 1 & 2\end{pmatrix}$

The following is an example for the BAP $\{5, 4, 6, 7, 8, 9\}$

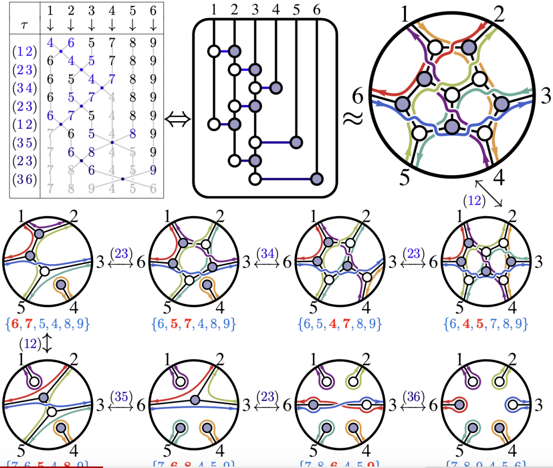


(figures from [ABCGPT])

BCFW Bridge Method

BCFW-Bridge Decomposition: Starting with any permutation σ, if σ is not a decoration of the identity, then decompose σ as $(a\,c) \circ \sigma'$ where $1 \leq a < c \leq n$ is the *lexicographically-first* pair separated only by legs b which are self-identified under σ and for which $\sigma(a) < \sigma(c)$; repeat until σ is the identity.

To illustrate this procedure, let's see how it generates a representative, reduced on-shell diagram which is labeled by the permutation $\{4, 6, 5, 7, 8, 9\}$:



BCFW Bridge Pipe Dreams

We can actually read off the sequence of BCFW bridges in the bridge decomposition method directly from pipe dreams.

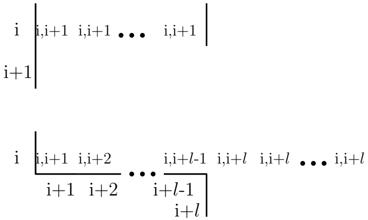
The rule is that we are always reading along southwest to northeast diagonals, starting from the northwest corner.

In other words, the word will be read in the order:

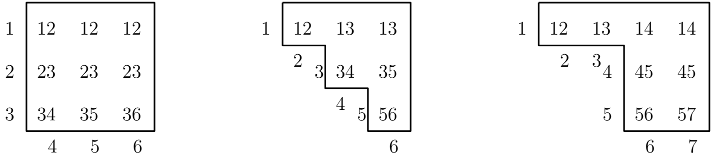
$$\boxed{\begin{matrix} 1 & 3 & 6 \\ 2 & 5 & 8 \\ 4 & 7 & 9 \end{matrix}}$$

In this example the shape was a square, but in practice the shape will be determined by the Grassmann necklace element I_1 of our choice of base

In terms of what letters or flips to fill in the squares of the diagram with, the rules are the following (1st rule can be seen as a special case of 2nd):



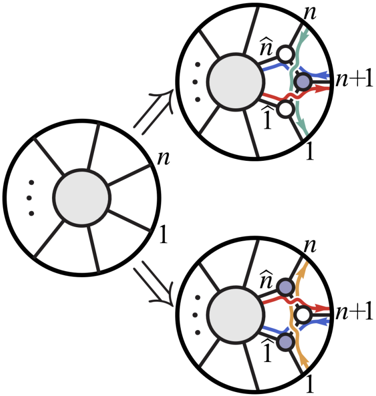
Here are some examples:



E.g. the 2nd example would consist of the sequence: 12,13,34,13,35,56

Inverse Soft Factors

From [ABCGPT]



k-preserving or *holomorphic* inverse-soft factor

momentum-space	momentum-twistors
$\lambda_{\hat n} = \lambda_n$ $\tilde\lambda_{\hat n} = \tilde\lambda_n - \alpha_{(n\,n+1)}\tilde\lambda_{n+1}$	$z_{\hat n} = z_n$
$\lambda_{\hat 1} = \lambda_1$ $\tilde\lambda_{\hat 1} = \tilde\lambda_1 - \alpha_{(1\,n+1)}\tilde\lambda_{n+1}$	$z_{\hat 1} = z_1$
$f(\cdots, n, n{+}1, 1, \cdots)$ $\Rightarrow f(\cdots, \hat n, \hat 1, \cdots)\times$ $\delta^2\big(\lambda_{n+1} - \alpha_{(n\,n+1)}\lambda_n - \alpha_{(1\,n+1)}\lambda_1\big)$	$f(\cdots, n, n{+}1, 1, \cdots)$ $\Rightarrow f(\cdots, n, 1, \cdots)$

k-increasing or *anti-holomorphic* inverse-soft factor

momentum-space	momentum-twistors
$\lambda_{\hat n} = \lambda_n + \alpha_{(n+1\,n)}\lambda_{n+1}$ $\tilde\lambda_{\hat n} = \tilde\lambda_n$	$z_{\hat n} = z_n + \alpha_{(n+1\,n)} z_{n-1}$
$\lambda_{\hat 1} = \lambda_1 + \alpha_{(n+1\,1)}\lambda_{n+1}$ $\tilde\lambda_{\hat 1} = \tilde\lambda_1$	$z_{\hat 1} = z_1 + \alpha_{(n+1\,1)} z_2$
$f(\cdots, n, n{+}1, 1, \cdots)$ $\Rightarrow f(\cdots, \hat n, \hat 1, \cdots)\times$ $\delta^2\big(\tilde\lambda_{n+1} + \alpha_{(n\,n+1)}\tilde\lambda_n + \alpha_{(1\,n+1)}\tilde\lambda_1\big)$	$f(\cdots, n, n{+}1, 1, \cdots)$ $\Rightarrow f(\cdots, \hat n, \hat 1, \cdots)$ $\times[n{-}1\; n\; n{+}1\; 1\; 2]$

Inverse Soft Factors

In many case, we can use inverse soft factors to build up an on-shell function (where Ω is the differential form obtained via a substitution for the supersymmetry variables)

We know from [He-Zhang] that that the form Ω for MHV and anti-MHV is of the following form:

$$\Omega_{n,2}^{\mathrm{tree}} = \prod_{i=1}^{n} \left(\frac{\langle i\, i+1 \rangle}{\langle 12 \rangle} \right)^{-1} \prod_{i=3}^{n} d\frac{\langle i2 \rangle}{\langle 12 \rangle} \wedge d\frac{\langle 1i \rangle}{\langle 12 \rangle} = \prod_{i=2}^{n-1} d\log \frac{\langle i\, i+1 \rangle}{\langle 1\, i+1 \rangle} \wedge d\log \frac{\langle 1\, i+1 \rangle}{\langle 1\, 2 \rangle}$$

The result for $\overline{\mathrm{MHV}}$ is the same with all λ's replaced with $\tilde{\lambda}$'s.

Inverse Soft Factors

and then we can build up more complicated forms by adding inverse soft factors:

$$k-\text{preserving}: \quad \Omega(\ldots,n,n{+}1,1) = \Omega(\ldots,\hat{n},\hat{1}) \wedge d\log\frac{\langle n\,n{+}1\rangle}{\langle n\,1\rangle} \wedge d\log\frac{\langle n{+}1\,1\rangle}{\langle n\,1\rangle}\,,$$

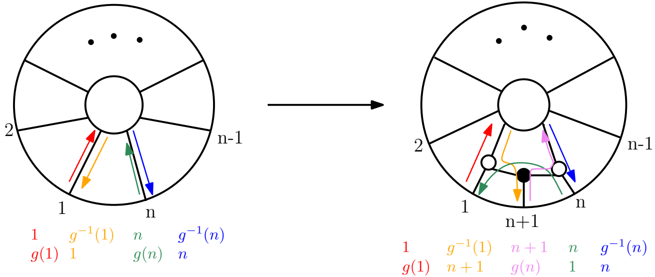
$$k-\text{increasing}: \quad \Omega(\ldots,n,n{+}1,1) = \Omega(\ldots,\hat{n},\hat{1}) \wedge d\log\frac{[n\,n{+}1]}{[n\,1]} \wedge d\log\frac{[n{+}1\,1]}{[n\,1]}\,,$$

where the first case $\tilde{\lambda}_{\hat{n}} = \tilde{\lambda}_n + \frac{\langle n+1\,1\rangle}{\langle n\,1\rangle}\tilde{\lambda}_{n+1}$, $\tilde{\lambda}_{\hat{1}} = \tilde{\lambda}_1 + \frac{\langle n\,n+1\rangle}{\langle n\,1\rangle}\tilde{\lambda}_{n+1}$ (λ's unchanged) and the second case $\lambda_{\hat{n}} = \lambda_n + \frac{[n+1\,1]}{[n\,1]}\lambda_{n+1}$, $\lambda_{\hat{1}} = \lambda_1 + \frac{[n\,n+1]}{[n\,1]}\lambda_{n+1}$ ($\tilde{\lambda}$'s unchanged).

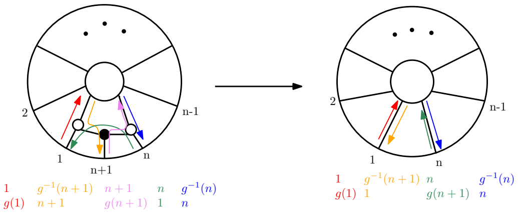
They note that if a permutation g for some particle i has $g(i-1) = i+1$ or $g(i+1) = i-1$, then i is a k-preserving or k-increasing inverse soft-factor, respectively.

Inverse Soft Factors

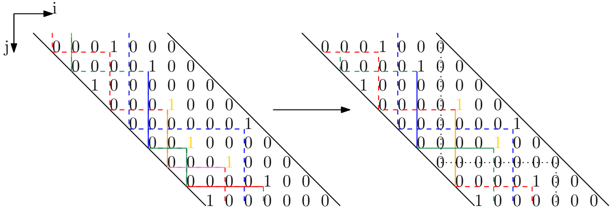
We can analyze what happens to the bounded affine permutation upon addition of an inverse soft factor:



However, it might be easier to see what's going on if we think about the removal of a soft factor:



And we can consider what happens on the diagonal diagrams, via an example (340362 from earlier)



Therefore, removal of a k-preserving soft factor is just deletion of i when $i-1$ has siteswap $f(i-1)=2$, so it can be done in two steps. Adding in a soft factor is the reverse of this. (The k-increasing soft factor is analogous, but where there is contraction instead)

The change for the bounded affine permutation upon addition of a k-preserving soft factor is:

(A) Tack a 0 on the end (that is, $f(n+1)=0$)
(B) Add $+1$ to every $f(i)$ that goes past $n+1$
(C) Change $f(n+1)$ to be $f(n)-1$
(D) Change $f(n)$ to be $f(n)=2$

Example

Let us put a soft factor on the end of sitedwap $f=22335$. We add 0 to the end: 223350. Add 1 to every number that goes past the 0 we just added: 223460. The last 2 number are 6 and 0; we switch the two, and change the $6 \mapsto 6-1=5$ and $0 \mapsto 2$. The final answer is 223425.

End

Thanks for listening!