**Research Statement**

Joseph Fluegemann

The Grassmannian $Gr(k,n)$ of $k$-planes in an $n$-dimensional vector space (usually $\mathbb{C}^n$ or $\mathbb{R}^n$) is a ubiquitous moduli space in mathematics. Schubert varieties provide a decomposition of the Grassmannian, and the study of these varieties has produced a great deal of interesting mathematics at the intersection of algebraic geometry, combinatorics, and representation theory. It turns out that there is a finer decomposition of the Grassmannian given by *positroid varieties*. Positroid varieties are connected to a lot of interesting combinatorics (see [Post] and [KLS11]) and even characterize certain quantum field theories [ABCGPT]. We include the definition of a positroid variety next for the unfamiliar reader:

Positroid varieties, denoted $\Pi_f$, are subsets of Grassmannia indexed by bounded affine permutations $f$ (which also describe certain types of juggling patterns, see [KLS11])

**Definition 1** (Bounded Affine Permutation)**.** *Fix $n \in \mathbb{N}$. A permutation $f : \mathbb{Z} \to \mathbb{Z}$ is called affine if it satisfies the periodicity condition $f(i+n) = f(i) + n$, $\forall i \in \mathbb{Z}$ .*

*If the affine permutation also satisfies $i \leq f(i) \leq i + n$ for all $i$, it is called bounded. We denote the set of bounded affine permutations of period $n$ and average jump $avg(g(i) - i) = k$ by $Bound(k,n)$.*

**Definition 2** (Bounded Affine Permutation Map)**.** *We can define a map $\tilde{f}$ from full rank $k$-by-$n$ matrices $M$ to $Bound(k,n)$: we take each column of a matrix, and assign the earliest next column for which it's in the span of the following:*

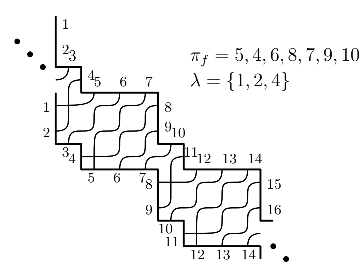
$$\tilde{f}_M(i) := \min\{j \geq i \colon \vec{c}_{i \bmod n} \in span(\vec{c}_{i+1 \bmod n}, \ldots, \vec{c}_{j \bmod n})\}.$$

**Definition 3** (Positroid Variety)**.** *For a given $f \in Bound(k,n)$, we can define the positroid variety $\Pi_f$ using the map $\tilde{f}_M(i)$ that we just described:*

$$\Pi_f = \{M \in Gr(k,n) \mid \tilde{f}_M(i) \leq f(i)\}$$

Much of my PhD work was devoted to studying whether certain special points (denoted $\lambda$) on positroid varieties are smooth or singular. These points of interest are the $\binom{[n]}{k}$ points fixed by the right torus action; these are precisely the points in the Grassmannian corresponding to the $k$-plane that is the span of $k$ coordinate lines, so $\lambda$ can be denoted by the set of $k$ coordinates entries.

There is a diagrammatic way of visualizing a positroid variety $\Pi_f$ in a neighborhood of $\lambda$ discovered in [Sni11]. These are called affine pipe dreams. The affine pipe dreams are composed of 2 types of tiles: elbows and crosses $+$, such that if one follows the lines (pipes) from the southwest border to the northeast border, one obtains the permutation $f$. The outer shape is determined by $\lambda$, which corresponds to the vertical line segments in the boundary. Here is an example:



We can alternatively think of each pipe dream as corresponding to a subword in a subword complex (the story is formalized in [KM03]). Informally, we can think of each of the squares in this diagram as being filled in with certain "letters," and then for each square in the pipe dream that has a cross (rather than an elbow), its letter gets "picked."

The main results in [Flu24] are:
(1) a correspondence between the degree of the tangent cone (therefore, smoothness vs. singularity) and the number of affine pipe dreams
(2) a characterization of what the affine pipe dreams that correspond to smooth points look like
(3) a simpler proof that, for the partial ordering composed of pairs $(\Pi_f, \lambda)$ with "deletion/contraction" (see end of this section) as the covering relation, smoothness of $\lambda$ on $\Pi_f$ is closed under going down, and a characterization of the generators of the order ideal (the minimal singular elements) in this ordering

Again, we provide the explicit statements here for the convenience of the reader.
(1)

**Theorem** (Smoothness Theorem)**.** *A positroid variety $\Pi_f$ is smooth at the point $\lambda$ (it meets $\lambda$ and is smooth there) if and only if there is a single affine pipe dream representative [Sni11] of the affine permutation $f$ on the open set $U_\lambda$.*

(2)

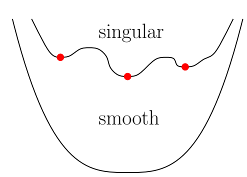
**Definition 4.** *A pipe dream in a rectangular shape* ***reduces to SE/NW partitions*** *if, after deleting all entire rows and columns of crosses, the result is a partition in the southeast and northwest corners.*

**Theorem** (Smooth Point Characterization)**.** *If each maximal rectangle in the pipe dream for $\Pi_f$ on $U_\lambda$ reduces to SE/NW partitions, then $\Pi_f$ is smooth at $\lambda$; otherwise, it is singular.*

(3)

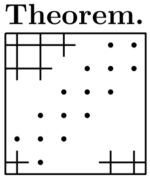
**Proposition 1.** *If $\lambda$ is smooth on $\Pi_f$, then $del_i(\lambda)$ is smooth on $\Pi_{del_i(f)}$ and $contr_i(\lambda)$ is smooth on $\Pi_{contr_i(f)}$.*

In other words, smooth positroid varieties remain smooth under deletion and contraction. This means that in the ordering (given by deletion and contraction) of positroid varieties and points, the smooth versus singular pairs can be divided in the following way:




The generators of the order ideal are depicted in the red dots above, and are called *atomic positroid pairs.*

**Theorem.** *The atomic positroid pairs have affine pipe dreams that look like the following:*



*More precisely: (1) We are in $Gr(k, 2k)$ and $\lambda$ consists of $k$ consecutive columns. (2) There is a single cross along the main SW to NE diagonal, with the diagonals adjacent to this diagonal having no crosses. (3) Other than (2), the only crosses exist in partition in the northwest and southeast*

Lastly, since deletion and contraction will be important for the following sections, we briefly describe what these operations are from a more general theoretical point of view.

We are in some flag variety (such at the Grassmannian) $Flag$ with a right torus $T$ action. Suppose we have a subgroup $S$ of $T$. Clearly, since $S \subseteq T$, all points $x$ that are fixed by $T$ are also fixed by $S$: $Flag^T \subseteq Flag^S$. There are certain properties "PPT" such that if $M$ be an irreducible subvariety of $Flag$ that is "PPT" at $x$, then the fixed points in $M$ under the action of $S$ ($M^S$) are also "PPT" at $x$. Smoothness and Cohen-Macaulay are in this class of properties "PPT." Deletion/contraction is for the case when the group $S$ is the circle in the torus given by $diag(1, ..., 1, z, 1, ...1)$. Let $M$ be a positroid variety. Note that $M^S$ is not irreducible - it is reducible and, in fact, disconnected: it consists of two disjoint sets, which are called the deletion and contraction: $M^S = Del_i(M) \sqcup Contr_i(M)$. Furthermore, since PPT (say, smoothness) is local, disconnected components are not relevant, so we can analyze the pieces separately.

Next, I discuss several interesting questions we can ask to try to extend the results above in new directions.

### Schubert Varieties in a Flag Variety

First, we can try modifying the mathematical objects to consider Schubert varieties $X_v$ and points $\mu$ in a finite-dimensional flag variety $Flags_n$. Much of the machinery used in [Flu24] carries over to this mathematical setting. We again can consider words $Q(i)$ that represent a point $\mu$ and study subwords of $\mu$ that represent $v$; this allows us to do analysis similar to that of the affine pipe dreams above, with the goal of finding a characterization of smooth points on Schubert varieties.

There does already exist a criterion for smoothness of a Schubert variety at a point which involves looking at reflections using positive roots to determine the size of the tangent space, which is described in more detail in the introduction of [Flu24]. However, a combinatorial method using subwords could be a topic of future investigation.

I already made a small step in this direction in [Flu24]. In the case when $v$ is 321-avoiding, the analysis for positroid varieties carries over straightforwardly, and is the subject of the following two propositions in that paper (Propositions 13 and 14 in the Arxiv version):

**Proposition 2.** *Let $\pi \in S_n$. The following are equivalent:*

*(1) $\pi$ is 321-avoiding.*
*(2) $T \curvearrowright X_0^\pi$ contains dilation.*
*(3) $\pi$ has a unique reduced word up to commuting moves (a unique heap).*
*(4) The heap is a skew partition.*

**Proposition 3.** *Let $Fl(n)$ denote the variety of flags in $\mathbb{C}^n$. Let $w \in S_n$ denote a permutation on $n$ elements so that $X_w$ denotes a Schubert variety in this flag manifold, and let $v \in S_n$ be 321-avoiding with $v \geq w$ in Bruhat order. Then $X_w$ is smooth at $v$, if and only if there exists a pipe dream for $w$ inside $v$'s skew partition (defined above Prop 2) such that all maximal rectangles reduce to NW/SE partitions. (In this case, the pipe dream will be unique.)*

Recall the discussion of deletion and contraction above. This setup for the positroid case carries over directly to the Schubert case because we can again consider the circle action $S$ and the fixed-points components under this action, which will consists of smaller Schubert varieties and points on them. Since the closure of smoothness under going down in the ordering still holds, we can ask a similar question about a characterization of the analogue of "atomic pairs" - generators of the upwards order ideal of singular pairs.

As explained above, in the positroid case, one can test smoothness by taking the unique reduced word for $\lambda$ and looking for subwords of it corresponding to a positroid variety $\Pi_f$. We cannot do this for Schubert varieties in a flag manifold because points $\mu$ can in general have multiple reduced words representing them, so there is no unique reduced word for $\mu$. We can attempt to remedy this by considering the following theorem:

**Theorem 1.** *If we have a Schubert variety $X_v$ containing a point $\mu$ that has some reduced word such that $v$ has one unique reduced subword of $\mu$, then $X_v$ is smooth at $\mu$.*

However, the converse has not yet been proved. We can conjecture (in type A):

**Conjecture 1.** *If $X_v$ is smooth at $\mu$, then there exists some word $\prod Q(i)$ for $\mu$ such that there is a unique reduced subword for $v$ inside $\prod Q(i)$.*

The reason why we can bypass this in the case of 321-avoiding permutations $\mu$ is because of Prop 2, which says that $\mu$ has a unique heap, which is a skew-partition. In the case of positroid varieties, the words for $\lambda$ are always skew-partitions, so are also unique (up to commuting moves).

Note that Theorem 1 is already sufficient basis to start looking at reduced words (for more general non-skew-parition shapes) and trying to obtain some sort of characterization of the smooth pairs. It would be interesting to see what kinds of shapes for the points do not allow for unique subwords for the Schubert variety in contrast to the ones that do. However, to harness the full power of the results in [Flu24], we would really like to have the bijective statements concerning smoothness and subwords, so proving the conjecture above is important.

## Singularity Structure

In the work in [Flu24], a determination of singularity vs. smoothness was only at the $\binom{[n]}{k}$ fixed points $\lambda$. The question of whether a given positroid variety $\Pi_f$ has any singularity was answered in [BW22] (although Theorem 2 in [Flu24] notes that if $\Pi_f$ has any singularity, then one of the points $\lambda$ must be singular.) However, the structure of the singular locus is still an open question.

We do know a few things about the singularity structure of on positroid varieties. First, because positroid varieties are normal, singularities are codimension 2 or higher [KLS11]. In addition, we have the following proposition:

**Proposition 4.** *For a given positroid vareity $\Pi_f$, the singular locus $Sing(\Pi_f)$ of $\Pi_f$ is a union of smaller positroid varieties: $Sing(\Pi_f) = \cup_i^m \Pi_{f_i}, \forall \Pi_{f_i} \subseteq \Pi_f$*

This follows from the local picture in [Sni11]. In a bit more detail, [Sni11] gives gives a stratified isomorphism between positroid varieties and Schubert varieties on an open chart. Since the singular locus of a Schubert variety is a union of Schubert subvarieties [BL00], this implies that the singular locus of a positroid variety is also union of positroid varieties.

One can think of a couple of small steps one can take, building on [Flu24], to start to unravel the singularity structure.

For example, we can try to determine whether a singular $T$-fixed point is an isolated singularity. We have the following rough criterion, which provides a sufficient (but probably far from necessary) condition for a point to be an isolated singularity.

**Proposition 5.** *Take a positroid variety $\Pi_f$. Consider a torus-fixed point $\lambda \in Gr(k,n)$. This point can be written by picking $k$ columns $\{c_1, ...., c_k\} \in \binom{[n]}{k}$ to be identity columns, with all other columns being zero columns. These zero columns are $\{d_1, ..., d_{n-k}\} = [n] \backslash \{c_1, ...., c_k\}$. Remove one $c_i \in \{c_1, ...., c_k\}$ and replacing it with one $d_j \in \{d_1, ..., d_{n-k}\}$, resulting in something like $\{c_1, ...., \hat{c}_i, ..., c_k, d_j\}$. Let's label as $\{\mu\}$ the set of all possible $\{c_1, ...., \hat{c}_i, ..., c_k, d_j\}$. If $\lambda$ is singular on $\Pi_f$, but $\{\mu\}$ consists entirely of smooth points on $\Pi_f$ (or empty), then $\lambda$ is an isolated singularity.*

One interesting thing we can do in this direction is to consider the $\mathbb{P}^1$ running between two points $\lambda_1$ and $\lambda_2$ that are related by the sort of switching of a single $c_i$ and $d_j$ that we described above (i.e. $\lambda_1 = \{c_1, ...., c_k\}$, $\lambda_2 = \{c_1, ...., \hat{c}_i, ..., c_k, d_j\}$) This $\mathbb{P}^1$ is formed by taking the $k-1$ coordinate lines that are common to both points (e.g. $\{c_1, ...., \hat{c}_i, ..., c_k$ in the notation above), which spans a $k-1$-plane; we then obtain the $k$-plane by adding in any vector in the $c_i - d_j$-plane.

If either points $\lambda_1$ or $\lambda_2$ are smooth, then the $\mathbb{P}^1$ between the 2 points is generically smooth (this follows from the fact that the singular locus is torus-invariant and closed). However, a first question we can ask is whether it is possible when both points are singular, for the $\mathbb{P}^1$ to be smooth, and what we can say about when this is the case. We could start by computing the equivariant cohomology of the restriction of $\Pi_f$ to these two points and using this calculation to help make sense of how to relate the affine pipe dreams between the two points.

Rather than the partially ordered set (positroid variety, point), we can consider (positroid variety, copy of $\mathbb{P}^1$), where the covering relation is again deletion or contraction, but in this case, the columns $c_i$ and $d_j$ cannot be deleted or contracted without removing the copy of $\mathbb{P}^1$. By the statement above, as soon as one of the two points becomes smooth, the $\mathbb{P}^1$ becomes smooth as well, but it could be the case that it becomes smoother even higher in the ordering. Finding a characterization of the minimal singular elements in this partial order could be an interesting question as well.

### Physics

Another direction in which I've been thinking recently is related to theoretical physics. Positroid varieties are connected to a branch of theoretical physics called "Amplitudes," since the goal is to calculate the scattering amplitudes of quantum field theory using new techniques that are different than the traditional methods of Feynman diagrams. I have been involved in this community since participating in a class led by IAS theoretical physicist Nima Arkani-Hamed in 2019 and even been an organizer for an Amplitudes journal club (other details are in my C.V). The foundational material connecting positroid varieties to this field of physics can be found in [ABCGPT] and [EH]. I have summarized many of the key ideas, offering my own perspective and filling in some gaps, in my thesis [FluThesis].

A very abbreviated summary is that positroid varieties come into play when calculating the leading singularities of planar maximally ($\mathcal{N} = 4$) supersymmetric Yang-Mills theory. The leading singularities are capture by the following "Link Formula," where the "C-matrix" gives a parameterization of a positroid variety:

$$f_\sigma^{(k)} = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_d}{\alpha_d} \delta^{k\times 4}(C \cdot \tilde{\eta}) \delta^{k\times 2}(C \cdot \tilde{\lambda}) \delta^{2\times(n-k)}(\lambda \cdot C^\perp)$$

I tried to connect the work in [Flu24] with the physics. In the last part of my thesis (to hopefully be reorganized into a new paper soon), I made the following connections:

(1) The $\alpha$'s in the Link Formula above are obtained via a canonical method of putting coordinates on a positroid variety called the "BCFW bridge decomposition method" (see section 5.2 of [ABCGPT]). The affine pipe dreams in [Flu24] can be used to "read off" the sequence of bridges in a pictorial/diagrammatic way.

(2) One can view inverse soft factors (see section 16.2 of [ABCGPT] and [HeZh]) as a way of building up plabic diagrams (a class of diagrams which are in correspondence with positroid varieties, described in [Post]). There is a way to view the amplitude as a differential form (by making a small change to the $\eta$ variables in the Link Formula, so then each positroid variety produces a differential form. This differential form is not easy in general to write down, but can be obtained

if one knows how to build up the plabic diagram via a series of inverse soft factors. I describe how to obtain the bounded affine permutation for a series of inverse soft factors. Furthermore, by using what I call *diagonal diagrams*, it is easy to see the inverse soft factors as a specific type of operation that is undone by deletion and contraction.

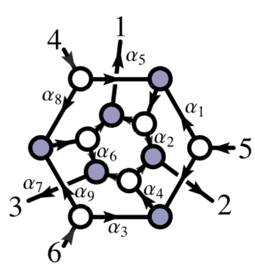
Finally, I have been thinking about how to apply deletion and contraction to gain a better understanding of *nonplanar on-shell diagrams*, which I will call *nonplabic diagrams*. These are like plabic diagrams, in that they are bicolored and can be drawn with entirely trivalent vertices, but they cannot be drawn without edges crossing each other if all "external edges" are on the boundary of a disc.

One of the cases that is very interesting for the physics is the 6-point NMHV case, corresponding to $Gr(3, 6, \mathbb{R})$. By calculating a very large number of examples, [BFGW] determined that the nonplabic diagrams produce 24 inequivalent top-dimensional "varieties" in $Gr(3, 6, \mathbb{R})$. However, these are open sets. We conjecture that they are all of $Gr(3, 6, \mathbb{R})$ complemented by a divisor, which is the vanishing locus of a section of the anticanonical bundle.

These nonplabic diagrams are difficult to deal with (they have a number of moves that preserve the equivalence class, including so-called "sphere moves" ([CEGM]) and possibly others). It may be best to start working with the varieties themselves, for which we have a $C$-matrix parameterization obtained by using the boundary measurement matrix after picking a perfect orientation [Post]. Deletion and contraction of plabic diagrams is not a nice story (we know how it works on pipe dreams, but the map from pipe dreams to plabic diagrams does not interact nicely with moves). I describe in [FluThesis] how deletion and contraction work on the C-matrix.

We know that deletion takes $Gr(3, 6)$ to $Gr(3, 5)$. Via Grassmannian duality (which can be obtained on the plabic or nonplabic diagrams by switching the white and black vertices), $Gr(3, 5)$ is isomorphic to $Gr(2, 5)$, so we can map to a variety within $Gr(2, 5)$. Contraction takes $Gr(3, 6)$ to $Gr(2, 5)$. Thus, in both cases, we get a map to (something isomorphic to) $Gr(2, 5)$. Now, the $Gr(2, n)$ case is the MHV (maximally helicity violating case). We understand the nonplanar MHV amplitudes [ABCPT14]: the differential forms consists of sums of positroid varieties with rotated coordinates. Thus, the question is whether the deletions and contractions give us MHV amplitudes. If so, what is the map? And can it shed light on the geometry of nonplanar NMHV amplitudes? One of the difficulties will be determining when a $C$-matrix corresponds to a nonvanishing rational function for generic momenta, since this is the case for which the nice results of [ABCPT14] hold.

We do a quick example here. Let's take the top-cell for the (planar) positroid variety in $Gr(3, 6)$. A representative on-shell diagram (with a perfect orientation) is provided by the following from [BFGW]:




$$\Longrightarrow C(\alpha) \equiv \begin{array}{c} \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{pmatrix} -\alpha_5(1+\alpha_8) & -\alpha_2 & \alpha_6\,\alpha_7\,\alpha_8 & 1 & 0 & 0 \\ \alpha_1\,\alpha_5 & \alpha_1\,\alpha_2+\alpha_4 & \alpha_4\,\alpha_7 & 0 & 1 & 0 \\ -\alpha_5\,\alpha_9 & \alpha_3\,\alpha_4 & \alpha_7(\alpha_3\alpha_4+\alpha_6\alpha_9) & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The contraction of column 5 gives us the vanishing set of the ideal $I = \langle a_1 a_5, a_1 a_2 + a_4, a_4 a_7 \rangle$, which has top-dimensional components $a_1 = a_4 = 0$. We conjecture that a point $\lambda$ exists on the open set parameterized by a C-matrix if and only if there is a perfect orientation such that all external edges corresponding to $\lambda$ point outwards.

Once we go to nonplabic diagrams, we start dealing with minors that are negative (e.g. see[PTZ]), so it makes sense to use oriented matroids to describe some of the combinatorial structure of these varieties. In this case, using an analogue of deletion and contraction for oriented matroids could uncover interesting results. The end goal would be to try to obtain some sort of combinatorial story for the varieties in $Gr(3,6)$ given by these nonplabic diagrams. If we are able to delete/contract these varieties, we could ask the additional question of whether we still have closure for smoothness under going down. One problem here is that the non-irreducibility of the contraction, as we saw in the example above, suggests that these varieties are not torus-invariant.

## References


[Post] A. Postnikov, Total positivity, Grassmannians, and networks, `math.CO/0609764`

[Flu24] J. Fluegemann, Smooth Points on Positroid Varieties, `math.CO/2407.21116`

[FluThesis] J. Fluegemann, Thesis: Smooth Points on Positroid Varieties and N=4 SYM Theory.

[KLS11] A. Knutson, T. Lam, and D. Speyer, Positroid Varieties: Juggling and Geometry, Journal? `math.AG/1111.3660`

[ABCGPT] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, and J. Trnka, Scattering Amplitudes and the Positive Grassmannian, `hep-th/1212.5605`

[Sni11] M. Snider, Affine Patches on Positroid Varieties and Affine Pipe Dreams (Thesis) `math.CO/1011.3705`

[KM03] A. Knutson and E. Miller, Subword Complexes in Coxeter Groups `math.CO/0309259`

[BW22] S. Billey and J. Weaver, A Pattern Avoidance Characterization for Smoothness of Positroid Varieties, Séminaire Lotharingien de Combinatoire XX (2022), Article #YY, 12 pp.

[BL00] S. Billey and V. Lakshmibai, Singular Loci of Schubert Varieties, 2000, Springer Progress in Mathematics.

[EH] H. Elvang and Y. Huang, Scattering Amplitudes in Gauge Theory and Gravity, Cambridge University Press, 1st edition 2015.

[HeZh] S. He, C. Zhang, Notes on Scattering Amplitudes as Differential Forms, `hep-th/1807.11051`

[ABCPT14] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Postnikov, and J. Trnka, On-Shell Structures of MHV Amplitudes Beyond the Planar Limit, `hep-th/1412.8475`

[BFGW] J. Bourjaily, S. Franco, D. Galloni, and C. Wen, Stratifying On-Shell Cluster Varieties: the Geometry of Non-Planar On-Shell Diagrams, `hep-th/1607.01781`

[CEGM] F. Cachazo, N. Early, A. Guevara, and S. Mizera, $\Delta$-Algebra and Scattering Amplitudes, `hep-th/1812.01168`

[PTZ] S. Paranjape, J. Trnka, and M. Zheng. Non-planar BCFW Grassmannian Geometries, `hep-th/2208.02262`