Allen Knutson's Juggling Patterns and N=4 Supersymmetric Quantum Field Theory

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Beginning

What is a "theory"? Or more precisely, a quantum field theory? It's a Lagrangian, which is a function that encodes all the information about how things move/behave.
For example:
$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$.

This is a function of the single field ϕ with mass m. We will discuss the fields a little more in a moment. By assumption, to obey special relativity, there is some representation of the group $SO(1,3)$ acting on the terms which leaves the Lagrangian invariant (possibly up to total derivative).

Lagrangian cont.

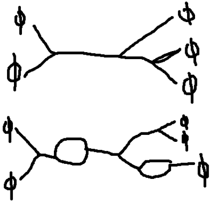
How should be understand the Lagrangian? Let's look ahead at how they'll be used in calculation:
$(\partial_\mu \phi)$ - this derivative and square is the typical kinetic term: it describes something like

The $m^2\phi^2$ says it has mass m: so a particle moving along freely with no interaction, which is not very interesting.

Interactions

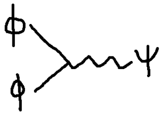
Since that wasn't very exciting, let's add a term to the Lagrangian: $L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + g\phi^3$, for some coupling constant g.
Now we have in interaction term with three ϕ fields:

Suppose we consider the process $\phi\phi \to \phi\phi\phi$. What processes are possible?
Here are some examples (the first is called a tree diagram, the second a loop diagram):



Interactions cont

We could also have interactions among different fields, e.g. something like:



Fields

Ok, we denoted objects like ϕ and called them fields, but what are they?
They are things like this:

$$\phi_0(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right),$$

The 0 subscript means free field. This is a scalar field because there is one entry: some fields have terms like $a_p^s u_p^s e^{-ipx}$ where the s denotes the basis vector u^s, and so there are called spinor or vector fields.
Why does it look like this?
Think about Fourier transforming the field $\phi(t, x) = \int \frac{d^3p}{(2\pi)^3} e^{ipx} \phi(t, p)$. Then apply the $O(1, 3)$ Laplacian $\Box$, and set equal to zero: $\Box\phi(t, x) = 0$ because this is actually the equation of motion (Euler-Lagrange equations) applied to the Lagrangian.

Fields cont.

The solution to this equation is given by $(\partial_t^2 - p^2)\phi(t,p)$: this actually has the same form as the harmonic oscillator equation from quantum mechanics (I know I'm skipping some steps here, sorry), so by analogy with quantum mechanics where we can write the position operator:
$x \sim a + a^\dagger$,
for creation/raising operators $a^\dagger$, and annihilation/lowering operators a (note: the operator a acts on the lowest-energy state in the Hilbert space $|0\rangle$ by sending it to zero: $a|0\rangle = 0$,

we can write the field:
$\phi \sim a_p + a_p^\dagger$.
Substituting it into the Fourier-transformed expression and doing a small minus-sign manipulation essentially produces the field expression.

Quantization

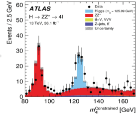
So far, we don't know what the symbols $a, a^\dagger$ are, but in perhaps the key step in all of this, we quantize. We impose the following relation, turning the a's into noncommuting operators:
$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p} - \vec{k})$
All the calculation happen from this relation so it is super important. Now, what if we add an interaction? We assume that everything is "perturbative" - i.e. the interactions are in a sense "small" compared to everthing else. The the field gets modified slightly:

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p(t)e^{-ipx} + a_p^\dagger(t)e^{ipx}\right]$$

While the $[a_k(t), a_p^\dagger(t)]$ relation is true at a single moment in time, now that the operators depend on time, they will rotate into each other. This means that it really helps for our calculations if we can write everything in terms of free fields.

Cross Section

Finally we can talk about scattering:
If you saw my poster for this talk, I showed the following plot, which gave evidence of the Higgs particle:



The expected and observed four-lepton invariant mass distribution for the selected Higgs boson candidates with a constrained Z boson mass, shown for an integrated luminosity of 36.1 fb −1 and at √ s = 13 TeV assuming the SM Higgs boson signal with a mass m H = 125.09 GeV.

The y-axis is a plot of $L \cdot \frac{d\sigma}{dm_{4L}}$, the only important part is the diffferential cross section: $d\sigma(\theta, E)$ in general depends on angles and energies. It tells us the probability that a scattering process will occur. We can write it as:

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v_1} - \vec{v_2}|} \left|M^2\right| \Pi_{\text{final states j}} \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_{p_j}} (2\pi)^4 \delta(\sum p)$$

Most of this is just experimental stuff that is the same for every experiment. The meat of the expression is all in the $|M|^2$ term.

Scattering Amplitude

M is called the "scattering amplitude". It is calculated via the S-matrix.
The S-matrix is defined as a matrix element for a process where initial states i (almost always 2 particle colliding) turn into states f, where i are assumed to be free at $t = -\infty$ and f at $t = \infty$.

We can write it as:
$\langle f|S|i\rangle_{Heisenberg} = \langle f; t = \infty|i; t = -\infty\rangle_{Schrodinger} =$
$lim_{t_1 \to -\infty, t_2 \to \infty} \langle f|e^{iH(t_2 - t_1)}|i\rangle$.
We can calculate S-matrices using the theoretical tools of Feynman diagrams. Actually, the scattering amplitude is really in the T-matrix:

$S = 1 + iT$, so we separate $\langle f|i\rangle$, a process that does nothing. The relation for the scattering amplitude is:
$\langle f|T|i\rangle = (2\pi)^2 \delta(\sum p)M$. The delta function imposes momentum conservation of the process.

S-matrix Calculations

Now, we can calculate the S-matrix using two big pieces of machinery:
The LSZ formula:

$$\langle p_3 \cdots p_n | S | p_1 p_2 \rangle = \left[i \int d^4 x_1 e^{-ip_1 x_1} (\Box_1 + m^2) \right] \cdots \left[i \int d^4 x_n e^{ip_n x_n} (\Box_n + m^2) \right] \times \langle \Omega | T\{\phi(x_1)\phi(x_2)\phi(x_3)\cdots\phi(x_n)\} | \Omega \rangle \,, \quad (6.19)$$

and writing it in free fields:

$$\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\} | \Omega \rangle = \frac{\langle 0 | \, T\left\{ \phi_0(x_1)\cdots\phi_0(x_n) e^{i\int d^4x \mathcal{L}_{\text{int}}[\phi_0]} \right\} |0\rangle}{\langle 0 | \, T\left\{ e^{i\int d^4x \mathcal{L}_{\text{int}}[\phi_0]} \right\} |0\rangle},$$

We plug the latter into the former. (I will not prove or show how to get these formulas since it might take a couple of lectures, but rather show how to think about them)

Feynman Diagrams 1

The LSZ formula

$$\langle p_3 \cdots p_n \,|S|\, p_1 p_2 \rangle = \left[i \int d^4 x_1 e^{-ip_1 x_1} (\Box_1 + m^2) \right] \cdots \left[i \int d^4 x_n e^{ip_n x_n} (\Box_n + m^2) \right] \times \langle \Omega | T\{\phi(x_1)\phi(x_2)\phi(x_3) \cdots \phi(x_n)\} | \Omega \rangle \,, \quad (6.19)$$

basically says take the thing in the second equation and then the integrals match up the particles with the data of the i and f in $\langle f|S|i\rangle$.

Feynman 2

The second formula:

$$\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\langle 0|\, T\left\{\phi_0(x_1)\cdots\phi_0(x_n)e^{i\int d^4x\mathcal{L}_{\text{int}}[\phi_0]}\right\}|0\rangle}{\langle 0|\, T\left\{e^{i\int d^4x\mathcal{L}_{\text{int}}[\phi_0]}\right\}|0\rangle},$$

gives us the Feynman rules/Feynman diagram expansion via a rule that is easy to prove called Wick's Theorem:
$T\{\phi_0(x_1)\cdots\phi_0(x_n)\} =: \phi_0(x_1)\cdots\phi_0(x_n) +$ all possible contractions:
The : stuff: symbol means that all operators a are moved to the right, and all $a^\dagger$ are moved to the left. This means that when they hit the vacuum $\langle 0|operators|0\rangle$, they die off, so only if everything is contracted do we get something nonzero.
What's a contraction? It basically comes from :
$[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p}-\vec{k})$, which is why we wanted to deal with free fields.

Feynman 3

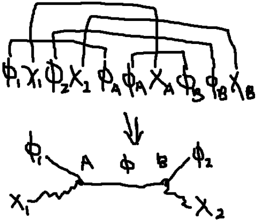
Let's close with an example that demonstrates how this works. Suppose $L_{int} = g\phi^2\chi$. We have $i = \phi_1, \chi_1$ and $f - \phi_2, \chi_2$.

If we look at
$\langle 0| T\{\phi_1\chi_1\phi_2\chi_2 e^{g\phi^2\chi}\}|0\rangle$
$= \langle 0| T\{\phi_1\chi_1\phi_2\chi_2(1 + g\phi^2\chi + \frac{1}{2}g^2\phi_A^2\chi_A\phi_B^2\chi_B + \cdots\}|0\rangle$

Note that since we are working perturbatively (so g is small), these terms become less and less significant. So let's cut off after the second power. Let's focus on the degree g^2 terms:
$\langle 0| T\{\phi_1\chi_1\phi_2\chi_2\frac{1}{2}g^2\phi_A^2\chi_A\phi_B^2\chi_B\}|0\rangle$

Feynman 4

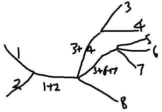
Denoting the contractions with a bar over the fields, one possibility is:



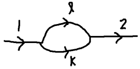
This diagram, according to the Feynman rules (which can be worked out by going through all the details of what I've described here), it will have a $\frac{g^2}{p_\phi^2 - m_\phi^2}$ for the 2 vertices and the propagator in the middle. All internal lines have this form of being in the denominator. This is just one diagram, for a single power of g, in the entire amplitude.

Loops

Final note: we need to make a distinction between tree-graphs and loop-graphs.
For tree graphs: by summing all the momenta, given knowledge of the external particles, everything is determined (on-shell):



However, for loop diagrams like:



the momenta are not all determined, so there will be an integral $\int d^4k$ in the amplitude. Thus we say that this particle is off-shell. The loop-diagrams are known for their difficulty, and require technqiues like renormalization. In what follows, everything will be on-shell, so we avoid the need to deal with off-shell diagrams.

Amplitudes Basics

Assume all particles massless.

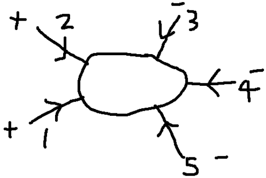
$p^\mu = \begin{pmatrix} p_1 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$. Rewrite as: $p_{ab} = \begin{pmatrix} -p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_0 - p_3 \end{pmatrix}$

$det(p_{ab} = p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0$ since massless.

Since the determinant is zero, we can decompose it as an outer product: $p_{ab} = p_1 \otimes p_2 = p\rangle[p = \lambda\tilde{\lambda}$ (different notation) (note $[pq], \langle pq\rangle$ are 2-by-2 determinants).

Amplitudes basics 2

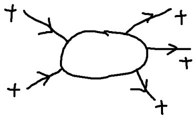
Take all particles incoming in our diagrams.
Particles characterized by mass and spin.
Mass is zero.
We deal here with helicity instead:
helicity=spin projected onto direction of momentum.
We can denote



as $A_5[++---]$

MHV

Because flipping direction of arrow flips the spin/helicity, we can redraw the previous diagram as:



So all positive helicities in an incoming to outgoing process.
This is a sort of intuitive reason why we always need at least two $+$'s or two -'s: $A[+++++] = A[+----] = 0$.

With exactly 2 minuses (called maximally violating helicity MHV amplitudes), we get the famous Parke-Taylor formula:
$A_n[--++\cdots+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$.
Famous since thousands of Feynman diagrams summed to this one term.

3-particle Amplitudes

For 3-particle amplitudes, there are 2 possibilities:

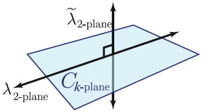
(MHV)−−+: $A_3[- - +] = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$

(anti-HMV)++-: $A_3[+ + -] = \frac{[12]^3}{[23][31]}$

We can depict this another way: we have two 2-planes:

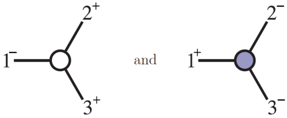
$\begin{pmatrix} 1]_a & 2]_a & 3]_a \\ 1]_b & 2]_b & 3]_b \end{pmatrix}$ and similarly for the $p\rangle$ momenta.

Momentum conservation says: $p_1 + p_2 + p_3 = 0$, which is $1\rangle[1 + 2\rangle[2 + 3\rangle[3 = 0$, which is a statement of a 2-by-2 matrix, which says that the planes spanned by the $p]$ matrix and $p\rangle$ matrix are orthogonal:



3-particle cont.

Since the planes are in 3D, either the $p\rangle$ 2-by-3 matrix is linearly dependent
$\Leftrightarrow$ $1\rangle \sim 2\rangle \sim 3\rangle$ are all proportional to each other
$\Leftrightarrow$ all the determinants $\langle ij\rangle$ vanish
$\Leftrightarrow$ we only have the $++-$ amplitude $= \frac{[12]^3}{[23][31]}$ nonvanishing.
We can depict this as:



where the white vertex is anti-MHV and the black vertex is MHV

BCFW (tree-level)

We shift two momenta i and j with a complex number.z as follows:

$[i \to [i + z[j, j\rangle \to j\rangle - zi\rangle$

So:

$p_i \to i\rangle[i + zi\rangle[j = p_i + zi\rangle[j$

$p_j \to j\rangle[j - zi\rangle[j = p_j - zi\rangle[j$

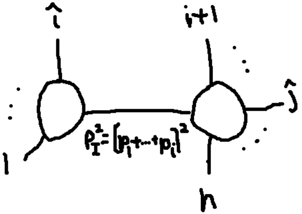
Now we think of the amplitude M as an analytic function of z, $M(z)$; the amplitude is $M(0)$.

BCFW

We look at $\frac{M(z)}{z}$ and aply Cauchy's Theorem (assumign no pole at infinity):

$0 = \oint \frac{dz}{2\pi i}\frac{1}{z}M(z) = M(0) + \sum_{\text{poles } z^*} \frac{1}{z^*} ResM(z^*)$

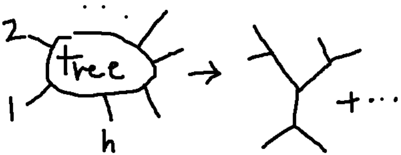
Remember form our Feynman diagrams that we have propagators $\frac{1}{p^2+m^2}$ on internal lines. Here we assumed that everythign is massless, so the poles are:



The pole only has z if i and j are on opposite sides, so the sum if $\sum_{z^*}$ is across all such lines.

BCFW recursion

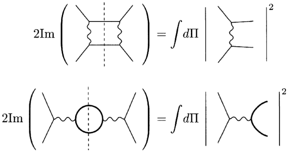
Thus, the BCFW formula says:
$M(0) = A_n = \sum_{\text{diagrams I}} \hat{A}_L(z^*) \frac{1}{P_I^2} \hat{A}_R(z^*)$.
Note that A_L and A_R are lower-point amplitudes, so the idea of this recursion is that we can rewrite the amplitude with lower-point amplitudes. Ideally, as is true for certain theories, we can break everything into products of 3-point amplitudes:



(Note that each sum has an implicit sum over all states that could be exchanged on the internal line, e.g. helicities $(-+), (+-)$.)

Optical Theorem/Unitarity Cuts/Cutkowsky

That last comment seems reminiscient of unitarity cuts. Diagrammatically:

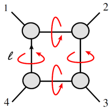


The $d\Pi$ is a sum across all states that can pass through the internal lines. Note that before we had to integrate over all momentum d^4p but now they are on shell: $p^2 = 0$, so we just have to sum across these states.

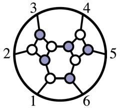
Optical theorem comes from a simple complex analysis exercise: $Im\frac{1}{p^2 - m^2 + i\epsilon} = -\pi\delta(p^2 - m^2)$ which produces the residue of the integrand at $p^2 = m^2$.

On-shell diagrams

In our cases of interest (i.e. N=4 SYM) this works for the full amplitude, not just the imaginary part.
So we can use maximal cuts on a diagram like:



to put all internal lines on shell (summed over possible internal states).
If everything is 3-particle vertices (which we can get by doing BCFW), then we get on-shell diagrams that look like:



On-Shell cont

We can think of on-shell diagrams in 3 ways:
(1) The aforementioned way of "maximal cuts" putting all propagators on-shell
(2) Taking 3-particle vertices and gluing them together, and attaching them with $\int d^2 I\rangle d^2 I] d^4\eta_I / U(1)$ (more on this method below)
(3) Taking an amplitude of a certain number of particles of certain helicities and applying BCFW continually until we've expanded the amplitude into a big sum of 3-particle vertex diagrams like the one shown in the previous page.

N=4 SYM

This is the reason why we don't need to label helicities on lines. Supersymmetry means that there is a symmetry of the Lagrangian of exchange of fermions (even-dim fields, half-integer spin/helicity) and bosons. So there are supersymmetry generators (Lie algebra generators of the Lie group).

N=4 SYM is a theory with 16 particles:

$$\underbrace{a}_{1\,\text{gluon}\,g^+},\quad \underbrace{a^A}_{4\,\text{gluinos}\,\lambda^A},\quad \underbrace{a^{AB}}_{6\,\text{scalars}\,S^{AB}},\quad \underbrace{a^{ABC}}_{4\,\text{gluinos}\,\lambda^{ABC}\sim\overline{\lambda}_D},\quad \underbrace{a^{1234}}_{1\,\text{gluon}\,g^-}$$

It is unique because it is maximally supersymmetry: every field can be moved to another field via applications of generators. Thus, we can write the entire thing as a superfield:

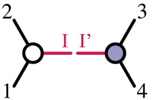
$$\Omega = g^+ + \eta_A\lambda^A - \frac{1}{2!}\eta_A\eta_B S^{AB} - \frac{1}{3!}\eta_A\eta_B\eta_C\lambda^{ABC} + \eta_1\eta_2\eta_3\eta_4\, g^-$$

And we have a notion of a superamplitude (curly A) with the superfield as inputs, from which we can project out the amplitudes:

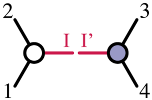
$$A_n(1^+\ldots i^-\ldots j^-\ldots n^+) = \left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{iA}}\right)\left(\prod_{B=1}^{4}\frac{\partial}{\partial\eta_{jB}}\right)\mathcal{A}_n(\Omega_1,\ldots,\Omega_n)\Big|_{\eta_{kC}=0}$$

Forming a Matrix

Each on-shell diagram is associated with a matrix. Recall (2), the idea of gluing together 3-particle amplitudes. The matrix is obviously associated to the momentum of each vertex, and then we glue:



$$\begin{array}{ccc} 1 & 2 & \mathrm{I} \\ \hline \end{array} \quad \begin{pmatrix} 1 & w_2 & w_{\mathrm{I}} \end{pmatrix} \qquad \begin{array}{ccc} \mathrm{I'} & 3 & 4 \\ \hline \end{array} \quad \begin{pmatrix} 1 & 0 & b_4^1 \\ 0 & 1 & b_4^2 \end{pmatrix}$$



$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \mathrm{I} & \mathrm{I'} & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & w_2 & w_{\mathrm{I}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & b_4^1 \\ 0 & 0 & 0 & 0 & 1 & b_4^2 \end{pmatrix} \end{array}$$

Direct/Outer Products

$(f_1, f_2) \mapsto f_1 \times f_2$

$(C_1, C_2) \mapsto C_1 \oplus C_2 \subset G(k_1+k_2, n_1+n_2)$

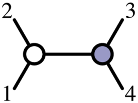
$(\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \Omega_2 \quad (d_1, d_2) \mapsto d_1+d_2$

Amalgamation: Gluing Legs (A, B)

$f \mapsto f' \qquad c_i \mapsto c_i \cap (c_A+c_B)^{\perp}$

$C \mapsto C/(c_A+c_B) \subset G(k-1, n-2)$

$\Omega \mapsto \Omega/\mathrm{vol}(GL(1)) \qquad d \mapsto d-1$



$$C \equiv \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & w_2 & 0 & b_4^1 \\ 0 & 0 & 1 & b_4^2 \end{pmatrix} \end{array}$$

Amplitude

That process is complicated and can get unwieldy very quickly. If we do manage to get a matrix, we can put it into this integral to get the amplitude (or rather, the part of the amplitude associated to this diagram):

$$f_\sigma^{(k)} = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_d}{\alpha_d}\, \delta^{k\times 4}(C\cdot\widetilde{\eta})\,\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp}).$$

This form can be found by going to the boundaries of the top-form of the Grassmannian:

$$f_\sigma^{(k)} = \oint_{C\subset\Gamma_\sigma} \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))}\, \frac{\delta^{k\times 4}(C\cdot\widetilde{\eta})}{(1\cdots k)\cdots(n\cdots k{-}1)}\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp}).$$

This integral was apparently the first place in the subject where Grassmannians entered. It was found by considering the most general formula obeying cyclic and Yangian symmetries. They found a $GL(k)$ symmetry of a $GL(k,n)$ matrix, which computes the amplitude for a process with k negative-helicity gluons and n total gluons, by taking residues around certain poles: each residue is a term in the BCFW expansion, and corresponds to an on-shell diagram.

Positroids

If we look at one of these matrices, we can compute a permutation: cyclically, the soonest column to the right where it is in the span of all columns to the right including this final column.

For example:

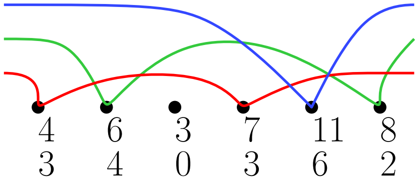
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

has permutation (3,4,0,3,6,2).

If we consider cutting up the Grassmannian by those matrices satisfying these sorts of permutation, then we get: the positroid stratification. These define nice varieties (positroid varieties) studied by Knutson, Lam, and Speyer.

Juggling Patterns

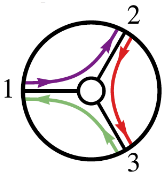
These permutations giving us positroid varieties actually correspond to jugging patterns:



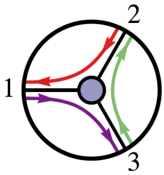
In $Gr(k, n)$, k is the number of balls, n is a period of say n seconds, and each time a column a is in the span of the following columns $a+1, ..., \sigma(a)$, is where the ball thrown at time a lands (at time $\sigma(a)$ seconds in front). Note these are one-hand juggling pattners.

Another Permutation

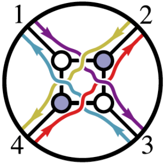
We have another permutation given by left-right paths: turn left at a white vertex, turn right at a black vertex:



$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix}$ and



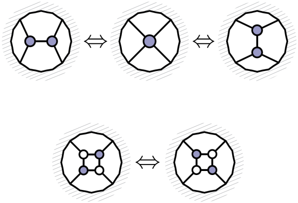
$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{pmatrix}$



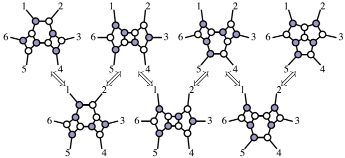
$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 1 & 2 \end{pmatrix}$

Left-Right Paths

One can show that these left-right paths encode the physics, because they are precisely the thing that is not changed by the following moves.

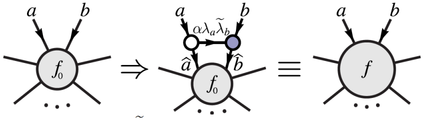


We can apply these to do something like the following:



Equivalence of Permutations

Let's show that these two permutations are the same. We do this via BCFW bridges. We show that the following insertion into a diagram is the same as doing BCFW (think (3) above):



Recall that for white vertices, all $p\rangle$ are parallel whereas all $p]$ are parallel for black vertices. The internal momentum is of the form $p\rangle[q$. Since it is attached to the first vertex where all $p\rangle$ are parallel to $p_a\rangle$, we must have $p\rangle = cp_a\rangle$ for some scalar c. Similarly for $[q = [p_b$.

Thus, adding the momenta together, we get:

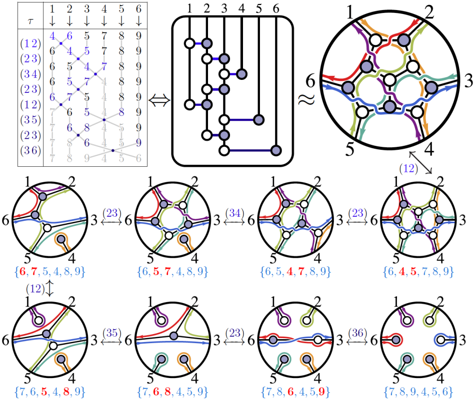
$\hat{p}_a = p_a + zp_a\rangle[p_b$

$\hat{p}_b = p_b - zp_a\rangle[p_b$

This is exactly BCFW!

BCFW bridges

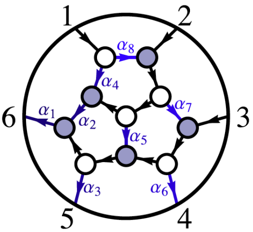
We can successively build up any permutation by using BCFW bridges. Here's an example: $\{4,6,5,7,8,9\}$. The rule is: $\sigma = (ac)\cdot\sigma'$ for $1 \le a < c \le n$ the lex-first separated pair with only self-b's in-between and $\sigma(a) < \sigma(c)$.



Equivalence (cont)

To show the equivalence between left-right paths and positroid permutations, we use the BCFW bridges to create a matrix that has the same permutation for both.

τ	$1 \downarrow$	$2 \downarrow$	$3 \downarrow$	$4 \downarrow$	$5 \downarrow$	$6 \downarrow$	BCFW shift
	4	6	5	7	8	9	
$(1\,2)$	6	4	5	7	8	9	$c_2 \mapsto c_2 + \alpha_8 c_1$
$(2\,3)$	6	5	4	7	8	9	$c_3 \mapsto c_3 + \alpha_7 c_2$
$(3\,4)$	6	5	7	4	8	9	$c_4 \mapsto c_4 + \alpha_6 c_3$
$(2\,3)$	6	7	5	4	8	9	$c_3 \mapsto c_3 + \alpha_5 c_2$
$(1\,2)$	7	6	5	4	8	9	$c_2 \mapsto c_2 + \alpha_4 c_1$
$(3\,5)$	7	6	8	4	5	9	$c_5 \mapsto c_5 + \alpha_3 c_3$
$(2\,3)$	7	8	6	4	5	9	$c_3 \mapsto c_3 + \alpha_2 c_2$
$(3\,6)$	7	8	9	4	5	6	$c_6 \mapsto c_6 + \alpha_1 c_3$



Matrix Cell

$$\underset{\{7,8,\mathbf{9},4,5,\mathbf{6}\}}{\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix}} \xrightarrow[\alpha_1]{(36)} \underset{\{7,\mathbf{8},\mathbf{6},4,5,9\}}{\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&\alpha_1\end{pmatrix}} \xrightarrow[\alpha_2]{(23)} \underset{\{7,6,\mathbf{8},4,\mathbf{5},9\}}{\begin{pmatrix}1&0&0&0&0&0\\0&1&\alpha_2&0&0&0\\0&0&1&0&0&\alpha_1\end{pmatrix}} \xrightarrow[\alpha_3]{(35)} \underset{\{\mathbf{7},\mathbf{6},5,4,8,9\}}{\begin{pmatrix}1&0&0&0&0&0\\0&1&\alpha_2&0&\alpha_2\alpha_3&0\\0&0&1&0&\alpha_3&\alpha_1\end{pmatrix}}$$

$$\underset{\{\mathbf{7},\mathbf{6},5,4,8,9\}}{\begin{pmatrix}1&0&0&0&0&0\\0&1&\alpha_2&0&\alpha_2\alpha_3&0\\0&0&1&0&\alpha_3&\alpha_1\end{pmatrix}} \xrightarrow[\alpha_4]{(12)} \underset{\{6,\mathbf{7},\mathbf{5},4,8,9\}}{\begin{pmatrix}1&\alpha_4&0&0&0&0\\0&1&\alpha_2&0&\alpha_2\alpha_3&0\\0&0&1&0&\alpha_3&\alpha_1\end{pmatrix}}$$

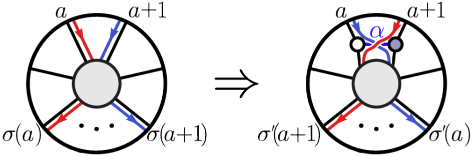
$$\underset{\{6,\mathbf{5},\mathbf{4},7,8,9\}}{\begin{pmatrix}1&\alpha_4&\alpha_4\alpha_5&\alpha_4\alpha_5\alpha_6&0&0\\0&1&(\alpha_2+\alpha_5)&(\alpha_2+\alpha_5)\,\alpha_6&\alpha_2\alpha_3&0\\0&0&1&\alpha_6&\alpha_3&\alpha_1\end{pmatrix}} \xleftarrow[\alpha_6]{(34)} \underset{\{6,5,\mathbf{7},\mathbf{4},8,9\}}{\begin{pmatrix}1&\alpha_4&\alpha_4\alpha_5&0&0&0\\0&1&(\alpha_2+\alpha_5)&0&\alpha_2\alpha_3&0\\0&0&1&0&\alpha_3&\alpha_1\end{pmatrix}} \xleftarrow[\alpha_5]{(23)} \underset{\{6,\mathbf{7},\mathbf{5},4,8,9\}}{\begin{pmatrix}1&\alpha_4&0&0&0&0\\0&1&\alpha_2&0&\alpha_2\alpha_3&0\\0&0&1&0&\alpha_3&\alpha_1\end{pmatrix}}$$

$$\underset{\{6,\mathbf{5},\mathbf{4},7,8,9\}}{\begin{pmatrix}1&\alpha_4&\alpha_4\alpha_5&\alpha_4\alpha_5\alpha_6&0&0\\0&1&(\alpha_2+\alpha_5)&(\alpha_2+\alpha_5)\,\alpha_6&\alpha_2\alpha_3&0\\0&0&1&\alpha_6&\alpha_3&\alpha_1\end{pmatrix}} \xrightarrow[\alpha_7]{(23)} \underset{\{\mathbf{6},\mathbf{4},5,7,8,9\}}{\begin{pmatrix}1&\alpha_4&\alpha_4\,(\alpha_5+\alpha_7)&\alpha_4\alpha_5\alpha_6&0&0\\0&1&(\alpha_2+\alpha_5+\alpha_7)&(\alpha_2+\alpha_5)\,\alpha_6&\alpha_2\alpha_3&0\\0&0&1&\alpha_6&\alpha_3&\alpha_1\end{pmatrix}}$$

$$\underset{\{\mathbf{6},\mathbf{4},5,7,8,9\}}{\begin{pmatrix}1&\alpha_4&\alpha_4\,(\alpha_5+\alpha_7)&\alpha_4\alpha_5\alpha_6&0&0\\0&1&(\alpha_2+\alpha_5+\alpha_7)&(\alpha_2+\alpha_5)\,\alpha_6&\alpha_2\alpha_3&0\\0&0&1&\alpha_6&\alpha_3&\alpha_1\end{pmatrix}} \xrightarrow[\alpha_8]{(12)} \underset{\{4,6,5,7,8,9\}}{\begin{pmatrix}1&(\alpha_4+\alpha_8)&\alpha_4\,(\alpha_5+\alpha_7)&\alpha_4\alpha_5\alpha_6&0&0\\0&1&(\alpha_2+\alpha_5+\alpha_7)&(\alpha_2+\alpha_5)\,\alpha_6&\alpha_2\alpha_3&0\\0&0&1&\alpha_6&\alpha_3&\alpha_1\end{pmatrix}}$$

Equivalence Proof

The proof that this matrix has the same permutation for both left-right paths and the positroid consecutive-rows calculation is not difficult. The matrix was created by doing the following shift: $\hat{c_{a+1}} = c_{a+1} + \alpha c_a$. We have $\sigma(a) > \sigma(a+1)$. Since the shifted $\hat{c_{a+1}}$ now has a multiple of column c_a added, and the soonest column c_a is in the span of columns is $\sigma(a)$, this is the new value: $\sigma(\hat{c_{a+1}}) = \sigma(a)$. Similarly, rearranging we get $c_a = \frac{1}{\alpha}(\hat{c_{a+1}} - c_{a+1})$, since c_{a+1} is in the span of $\{c_{a+2}, ..., c_{\sigma(a+1)}\}$ and $\hat{c_{a+1}}$ is even before that, c_a is already in the span of $\hat{c_{a+1}}, ..., c_{\sigma(a+1)}$. That is, $\sigma(a) = \sigma(a+1)$. Thus, the BCFW bridge switches $\sigma(a)$ and $\sigma(a+1)$.



Summary

-That is, it is obvious diagrammatically that the BCFW bridges switch the left-right paths for the corresponding edges. What we have shown is that in does the same thing on the permutation obtained by considering linear dependencies of columns.

-This is in general a different matrix that found by fusing together 3-particle amplitudes. However, this doesn't matter because we saw that the physics is captured by the left-right paths.

-Thus, this matrix is sufficient for us to plug into the integral formula:

$$f_\sigma^{(k)} = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_d}{\alpha_d}\, \delta^{k\times 4}\big(C\cdot\widetilde{\eta}\big)\delta^{k\times 2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times(n-k)}\big(\lambda\cdot C^{\perp}\big).$$

Final

We have shown that juggling patterns characterize physics! (of planar on-shell N=4 SYM amplitudes)

Geometrically, juggling patterns are cells in the positroid stratification of the positive Grassmannian.

Each on-shell diagram, and the subset of matrices it determined via left-right paths, and the result of the integration, is a term in a BCFW expansion.

We can further decompose via BCFW by going to the boundaries of its positroid variety.