Smooth Points on Positroid Varieties

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Grassmannians

Let V be an n-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$)

For any integer $0 \leq k \leq n$, $Gr_k(V)$ is the set of all n-dimensional linear subspaces (through the origin) of V.

$Gr_k(V)$ can be given the structure of a smooth manifold of dimension $k(n-k)$

In this talk, we will be concerned with $Gr_k(\mathbb{C}^n)$ denoted $Gr(k,n)$.

Grassmannian Examples

Examples:

(1) $Gr(0, n) = \{0\}$ the origin

(2) $Gr(n, n) = \mathbb{C}^n$ the whole space (is a point in the Grassmannian)

We have a natural duality: $Gr(k, n) \cong Gr(n-k, n)$ by considering the "orthogonal complement" $L^\perp$ of $L \in Gr(k, n)$.

(3) $Gr(1, n) \cong \mathbb{P}^{n-1}$

(4) Thus by duality, $Gr(n-1, n) \cong \mathbb{P}^{n-1}$

Writing Grassmannians

We tend to write $Gr(k, n)$ as $GL(k)\backslash$[full rank $k \times n$ matrices], and think about it as k row vectors spanning a plane up to invertible basis transformation inside the plane.

Each point in $Gr(k, n)$ given by $GL(k)\backslash[k \times n]$, since it is full rank, has a set of columns $\lambda \in \binom{[n]}{k}$ whose determinant is nonzero, so we can use $GL(k)$ to set λ to be the identity so $\left(k \times n\right) \cong \begin{pmatrix} 1 & * & 0 & & 0 \\ 0 & * & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & 0 & & 1 \end{pmatrix}$ where the $00 \cdots 1 \cdots 0$ columns are in λ.

Grassmannian Cover

These $\binom{n}{k}$ choices of λ are all the fixed points of the Grassmannian under a right torus action. Furthermore, if we let the other column of $*$'s be anything, so that these denote open sets, then these open sets are isomorphic to $\mathbb{C}^{k(n-k)}$.

Furthermore, these open sets cover the Grassmannian. In other words, $Gr(k, n)$ has an open cover by affine patches (not just affine varieties but things isomorphic to $\mathbb{A}^n$ itself), and since any two patches have nonempty intersection, $Gr(k, n)$ is an irreducible variety of dimension $k(n-k)$.

We will work on these affine patches in the following, and we denote them by U_λ where λ refers to the k columns we have set equal to the identity (and by abuse of notation, also the point with all other columns equal to zero).

Definition of Positroid Variety

The way positroid varieties were initially defined was using Grassmann necklaces and cyclically rotated Schubert matroids. These are not difficult to deal with, but since they are tangential to our discussion in the following, we use this definition of positroid variety:

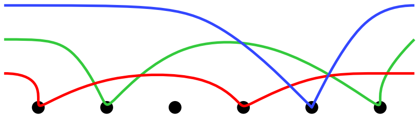
Definition (Positroid Variety)

$\Pi_f = GL_k \backslash \{M \subseteq M_{k,n} | rank([i,j]) \leq |[i,j]| - \#1's$ southwest of (i,j), $i \leq j \leq i+n\}$ where $rank([i,j])$ is defined cyclically

To understand and gain intuition for this definition, it is helpful to look at juggling patterns:

Positroid Varieties and Juggling

Here's an example of a juggling pattern:

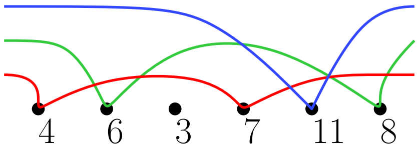


Each dot is a moment in time: the first dot is the time at the start, let's call it $t = 1$, the second dot is at $t = 2$

This juggling pattern repeats every 6 seconds.
There are 3 balls, one for each color.

We can think of the dots as the hand; note that this is a simplified model in which there is only one hand.

This looks somewhat like a permutation. We could describe it with the following list of numbers:



We have written below each point the moment in time when the ball thrown at that moment lands. So the first point has a 4 because the (red) ball thrown at $t = 1$ lands at time $t = 4$.

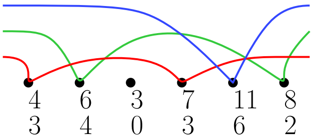
We can write it like: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix}$. We call this a bounded affine permutation.

Definition (Bounded Affine Permutation)

Affine permutation: $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the periodicity condition $f(i + n) = f(i) + n$, $\forall i \in \mathbb{Z}$ where n is the period.
A *bounded* affine permutation also satisfies: $i \leq f(i) \leq i + n$.

Siteswaps

But since this juggling pattern is periodic, there's no good reason for choosing red ball with the 4 for the first moment in time.
Instead of putting the time when each balls lands, we can make it permutation invariant by putting the units of time ahead of the present moment that the ball lands:



This second row of numbers is what we call the *siteswap*. Another way to see how we got this list of numbers is by taking the bottom row and subtracting the top row: $f(i) - i$ in the bounded affine permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix} \mapsto \begin{pmatrix} 4 & 6 & 3 & 7 & 11 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 \\ 3 & 4 & 0 & 3 & 6 & 2 \end{pmatrix}.$$

Now let's connect this to matrices and Grassmannians.

Here's the connection: a juggling pattern of length n with k balls corresponds to a set of k by n matrices.

Here's the map from full rank k-by-n matrices to siteswaps: we take each column of a matrix, and assign the earliest next column for which it's in the span of the following:
$f(i) := \min\{j \geq i : \vec{c}_i \in span(\vec{c}_{i+1 \bmod n}, \ldots, \vec{c}_{j \bmod n})\}$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$
which is a 4-by-n, has first number 3. Why? Because the first column is not in the span of the 2nd and 3rd columns, but only in the span of columns 2-4.

Positroid Variety from Siteswap

So now it should be clear how the map works. Clearly, there are a whole bunch of matrices that map to a given siteswap.

We have just given another definition of positroid variety:

Definition (Positroid Variety)

Positroid varieties, denoted Π_f are subsets of Grassmannia defined by the cyclic rank conditions imposed by siteswaps, or equivalently, bounded affine permutations f (f might also denote the siteswaps, but should be clear from context).

Example

As an example, we can take the siteswap considered above: $f = 340362$. The following matrix gives a point within Π_f:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let's now show how the first definition of positroid variety connects to this definition. The first definition involved some notion of "southwest 1's" and all cyclic rank conditions, so in particular we will show how to get it cyclic ranks $rank[i,j]$, not just $rank[i,f(i)]$, from the siteswaps.

First, let's switch back to doing bounded affine permutations. Our bounded affine permutation was $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix}$.

Let's put our bounded affine permutation into a matrix, and insert the first number:

$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$.

Let's do the second column as well:

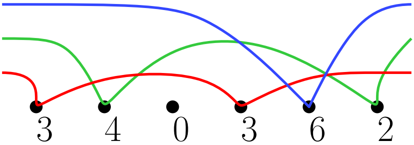
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that like a normal permutation matrix there is a single one in each row and column.

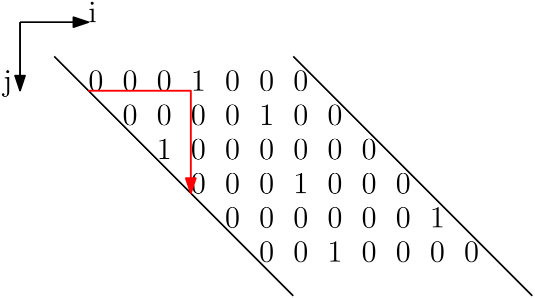
This is often times drawn with lines running from northwest to southeast showing the allowed region with horizontal axis i increasing to the right and vertical axis j increasing to the south:

$$\begin{array}{l} \longrightarrow i \\ \downarrow j \end{array} \qquad \begin{array}{cccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & & \\
 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & & & \\
 & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
 & & & & & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}$$

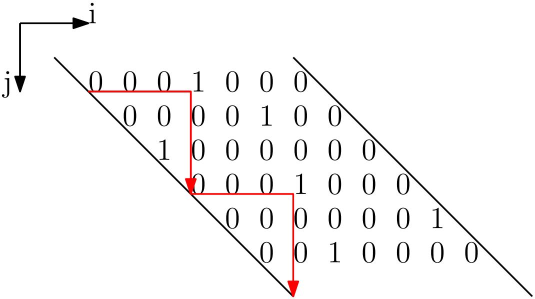
Let's recall our juggling pattern:



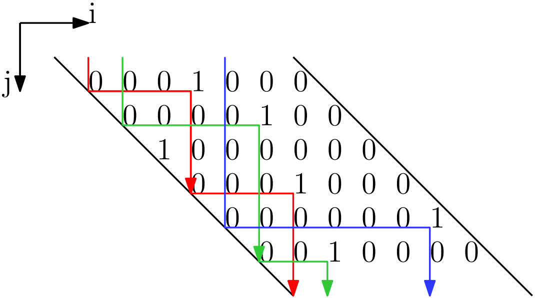
Let's draw the line on the permutation matrix corresponding to the first throw of the ball at time $t = 1$ that lands at time $t = 4$:



Let's draw the entire path of the ball in the juggling pattern represented by the red line:

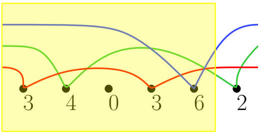


Let's show what the lines look like when we include all the balls in the juggling pattern:

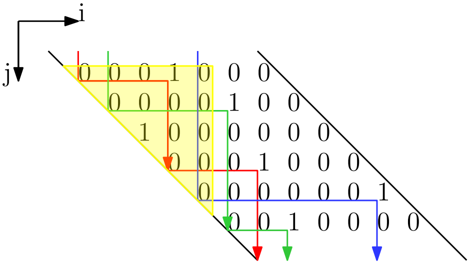


It's clear that what's going on is that we draw the lines horizontally from the left edge of the diagram until the line hits a 1, and then we go south until it hits the left edge of the diagram again, at which point we go horizontally right again.

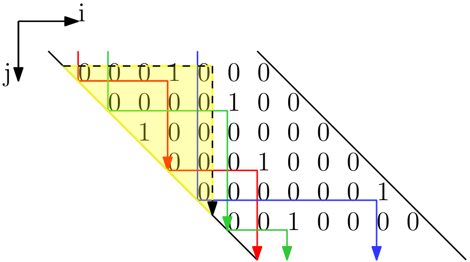
So finally, let's describe how we calculate the cyclic ranks. Let's do an example. Let's take the juggling pattern and let's look at $[i, j] = [1, 5]$:



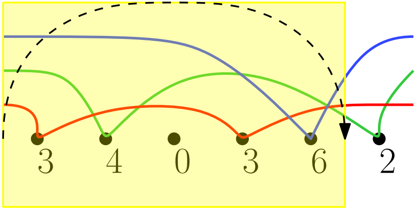
corresponding to the highlight in the permutation matrix:



There are two 1's in the highlighted portion: at (1,4) and (3,3). These are precisely the places of balls that are both thrown and landed within this interval $[1, 5]$.
Let's be clearer. Let the dotted black line in the permutation diagram be a hypothetical ball that would be thrown at time $t = 1$ and land at time $t = 5$:



This would correspond to a hypothetical ball of something like the dotted line below:



We have drawn it to start a little before time $t = 1$ and land a little after $t = 5$ to emphasize that it contains this interval (it includes both $t = 1$ and $t = 5$).

Definition of Positroid Variety with All Cyclic Ranks

Here's the formula for rank:

$rank[i,j] = |[i,j]| - \{\#1's \text{ to SW}\} = (j-i+1) - \{\#1's \text{ to SW}\}$. "SW" means "southwest."

This seems reminiscent of the first definition of positroid variety, reproduced below:

Definition (Positroid Variety)

$\Pi_f = GL_k \backslash \{M \subseteq M_{k,n} | rank([i,j]) \leq |[i,j]| - \#1's$ southwest of (i,j), $i \leq j \leq i+n\}$ where $rank([i,j])$ is defined cyclically

This should make sense from what we have just explained. In our example, $rank[1,5] = |[1,5]| - 2 = [(5-1)+1] - 2 = 3$. So the rank of this interval is 3.

This makes sense because given that the column when the ball was thrown is inside this yellow-highlighted region (and all columns up to and including when it lands are also inside the highlighted region), it is dependent on the other columns - so it should not increase the rank of the matrix.

There's another way we could have seen this from the diagram: the rank is the number of balls that enter (or leave - they're the same since balls are not created or destroyed) the interval. In our yellow region of the juggling diagram, we see that all three balls - red, green, and blue - all enter the region and exit.

And then it becomes clear why the 1's inside the highlighted triangle do not add to the rank: because these are ball-lines representing throws that are both made and land within this time interval, so they do not add to the number of lines entering and exiting the interval.

From Cyclic Ranks to Siteswap and Stratification

We've shown how to go from siteswap or bounded affine permutation to this matrix of cyclic ranks. How do we go in the other direction?

Answer: From the diagram of ranks, whenever we spot this pattern: $\begin{pmatrix} r & r \\ r & r-1 \end{pmatrix}$, it means that this is where a ball drops.

Another important fact is that positroid varieties form a stratification.

Definition (Stratification)

A stratification is a collection of subvarieties such that the intersection of any two is a union of others. Then we can talk about stratification generated by some subvarieties: if we repeatedly intersect and decompose them, then we get smaller subvarieties in the stratification.

Matroids

Matroids were created to capture the combinatorial essence of linear dependence. There are many characterizations/equivalent definitions. Here's one:

Definition (Matroid)

A matroid $M \subseteq \binom{[n]}{k}$ satisfies the exchange axiom:
For all $I, J \in M$ and all $i \in I$ there exists a $j \in J$ such that $(I \backslash \{i\}) \cup \{j\} \in M$

Example:
$n = 3, k = 2$
$I = \{1, 2\}, J = \{1, 3\}$
$i = 2 : I \backslash i = \{1\}$, and let $j = 3 \in J$, then $\{1\} \cup \{3\} = \{1, 3\} \in M$
$i = 1 : I \backslash i = \{2\}$, and let $j = 1 \in J$, then $\{2\} \cup \{1\} = \{1, 2\} \in M$

Realizable Matroids

We are specifically interested in realizable matroids: matroids whose elements are all the sets of columns of a matrix that give a nonzero maximal minor. We can describe such matroids as (E, B), a ground set E and bases B making up the matroid.

Example

We can get a realizable matroid from the matrix

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The ground set $E = \{1, 2, 3, 4, 5\}$ for the columns. This realizable matroid has the bases: $\{\{1, 4\}, \{1, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$

Realizable Matroids are Matroids

Theorem

Realizable matroids are matroids (satisfy the exchange axiom in the definition above).

We can take a matrix $A \in Gr(k, n)$, and we need to show that the columns of nonzero maximal minors do indeed satisfy the exchange relation given above.

Proof:
$I, J \in M \Leftrightarrow \Delta_I(A) \neq 0$ and $\Delta_J(A) \neq 0 \Leftrightarrow \Delta_I \cdot \Delta_J \neq 0$.
But we have $\Delta_I \cdot \Delta_J = \sum_{j \in J} \pm\Delta_{(J \backslash j) \cup i} \cdot \Delta_{(I \backslash i) \cup j}$ by the Plucker relations with one element i. Thus this sum must have at least one nonzero element which means there is a j such that $(I \backslash i) \cup j \in M$.

Positroids and Matroids

Definition (Matroid Strata)

For a given realizable matroid M, we can defined its matroid stratum S_M, defined as all points in $Gr(k, n)$ such that a matrix representative has nonvanishing maximal minor precisely for the bases in M and all other maximal minors equals to zero.

Definition (Positroid)

For a given $Gr(k, n)$, if we look at all matroid strata, and intersect with the positive real Grassmannian (all maximal minors real and positive), then the matroids whose strata are nonempty are called *positroids*

Positroids and Matroids (cont)

We will not prove this next theorem, but will state it for completeness of terminology:

Theorem

Positroids are in bijection with siteswaps. Furthermore, the bijection is that within the positroid variety Π_f of each siteswap f, there is a single dense matroid (positroid) stratum S_M contained within. They are not equal (in fact, the matroid strata do not form a stratification). However, their closures are equal. Hence, we can equivalently index the positroid varieties with positroids rather than siteswaps.

Questions

These are questions that were guiding my research recently. (Note: I will be defining these terms below. I will repeat this slide again later).

(1) We know geometrically why singular things are closed under going up in deletion/contraction, while smooth things are closed under going down. Can we prove this in a combinatorial way?

(2) Can we describe the atomic positroid varieties?

Definition (Atomic positroid variety)

A positroid variety is *atomic* if it is singular, but any deletion or contraction takes it to a smooth (or empty) positroid variety.

(3) How can we quickly test whether a positroid variety is singular at a given point?

Pipe Dreams

Definition (Pipe Dream)

A pipe dream in S_n is a diagram in an n-by-n square where each box can be filled by one of two types of tiles, elbows and crosses $+$, such that all crosses occur above the southwest-northeast diagonal. Thus, we can think of the grid as a set of pipes that begin on the north and east edges and end on the west and south edges (with the south to east tiles being all elbows)

We called the pipe dream *reduced* if none of the pipes cross twice. We will focus only on the reduced case in the following.

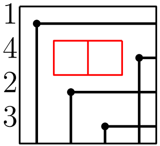
Next, we describe a couple of different and important ways to think about pipe dreams, focusing on examples.

Pipe Dreams - Rothe Diagram

Let's take a permutation of 4 elements, say: 1423.
Let's write the matrix corresponding to this permutation:

$$\pi_{1423} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & 1 & & \\ & & 1 & \end{pmatrix}.$$

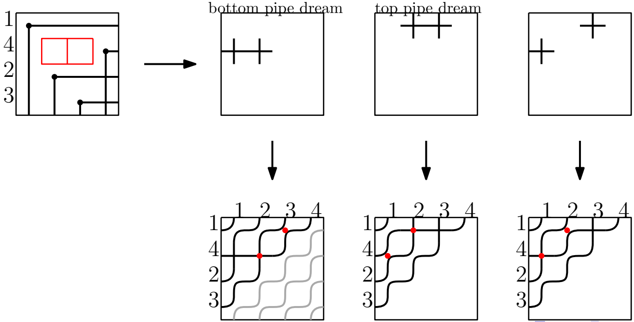
This matrix defines a matrix Schubert variety: all 4-by4 matrices M, such that the rank of the $p \times q$ northwest submatrix, denoted $r_{pq}(M)$ is less than or equal to that of π_{1423}: $r_{pq}(M) \leq r_{pq}(\pi_{1423})$.
We can draw something called a Rothe diagram associated with π_{1423}, given by drawing lines south and east from each 1 in the permutation matrix:



Pipe Dreams - Rothe Diagram (cont)

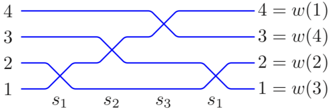
This way of thinking about things is connected to the dimensionality of the matrix Schubert variety: the number of boxes is the number of conditions imposed, equivalently the number of inversions or length of the shortest word (see below), which drop the dimension.

If we turn the boxes into crosses and push them to the left, we get the "bottom pipe dream"; to the top is the "top pipe dream." The red dots in the figure below show a potential "move": a crossing and then near-miss (important for what follows).



Wiring Diagrams

Another way to think about pipe dreams is as wiring diagrams. We review what wiring diagrams are on this slide. Here is an example from Postnikov:



In this context, reduced means that no two lines cross twice. In this example, if we trace the pipes from left to right, we get the permutation 4213. With transpositions $(i, i+1)$ denoted by s_i, we see that the wiring diagram gives us a word $s_1 s_2 s_3 s_1$ which produces this permutation:

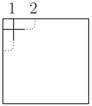
$$s_1 s_2 s_3 s_1(1234) = s_1 s_2 s_3(2134) = s_1 s_2(2143) = s_1(2413) = (4213).$$

For these words, reduced corresponds to it being the shortest word producing a given permutation.

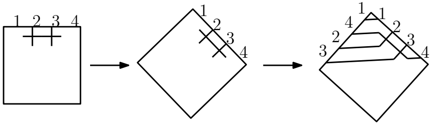
Pipe Dreams - Wiring Diagrams

Now let's see what the transpositions are in the square from before:

$$\begin{array}{|cccc|}\hline s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \\ s_4 & s_5 & s_6 & s_7 \\ \hline \end{array}$$



We read from top to bottom and left to right. The second figure shows that if we just had the northwesternmost cross, it would switch pipes 1 and 2, so it makes sense for it to be s_1. For the example above of 1423 it's clear that all pipe dreams give the same word $s_3 s_2$. We can thus view the pipe dream as a rotated wiring diagram producing a reduced word:

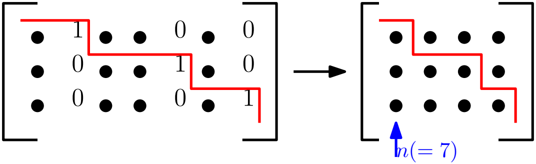


Here we are reading right to left, so in the order opposite to Postnikov, we can verify: $s_3 s_2(1234) = s_2(1243) = 1423$.

Affine Pipe Dreams

Recall that we are studying positroid varieties on the open affine patches U_λ, these open neighborhoods of fixed points λ. We have an analogue of pipe dreams for bounded affine permutations called *affine pipe dreams* invented by Allen Knutson's previous student Michelle Snider. We do an example to show how this works.

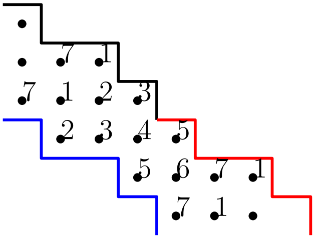
Let's take an example of a siteswap 2225377 (bounded affine permutation $34598, 13, 14$) inside Gr(3,7), and let's test it on $\lambda = \{2, 5, 7\}$.
We first write the U_λ corresponding to the point $\lambda = \{2, 5, 7\}$:



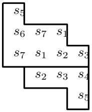
Next, we (1) redraw this line shifted up and right, creating a finite strip (2) label the bottom line with the column number, and the top line with the corresponding number. We thus see what the transpositions should be.

In analogy with the usual case, they run diagonally southwest to northeast, with s_1 being where we can switch pipes 1 and 2.

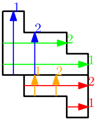
Filling in the rest, we get:



If we just look at one block, denoting the transpositions by s_i again, it is:



To get the bottom pipe dream, we have to pick a way of reading this. There are equivalent ways of reading it, given by:



In other words, if we choose the direction of red followed by green, then the ordering will be $s_5, s_2s_3s_4, s_7s_1s_2s_3, s_6s_7s_1, s_5$. If we apply this word to $(k+1, k+2, ...k+n)$, then we get the bounded affine permutation correspond to $+n$ for the λ columns $2, 5, 7$ and $+0$ for all other columns. This shows that this pipe dream shape represents the neighborhood U_λ.

Here's how reading it this way produces a word for siteswap 2225377 (bounded affine permutation $34598, 13, 14$):

We start with:

1	2	3	4	5	6	7	8(=1)
3	4	5	9	8	13	14	

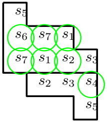
Since the big word is $s_5, s_2 s_3 s_4, s_7 s_1 s_2 s_3, s_6 s_7 s_1, s_5$, we find the first one where there was a transposition with $f(i) > f(j), i < j$. This first one is s_4 which switches the 98 to 89:

1	2	3	4	5	6	7	8(=1)
3	4	5	9	8	13	14	
3	4	5	8	9	13	14	10

We continue on until we get the most general permutation:

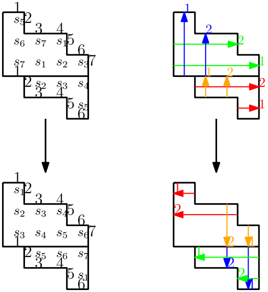
1	2	3	4	5	6	7	8(=1)
3	4	5	9	8	13	14	
3	4	5	8	9	13	14	10
7	4	5	8	9	13	10	14
4	7	5	8	9	13	10	11
4	5	7	8	9	13	10	11
4	5	7	8	9	10	13	11
6	5	7	8	9	10	11	13
5	6	7	8	9	10	11	12

Keeping track of which transpositions we used, we get:



If we put crosses where the circles are, and elbows elsewhere, then we get the bottom pipe dream.

The top pipe dream corresponds to the bottom pipe dream that would obtain if we used Grassmannian duality and reflected the pipe dream across the northwest to southeast line. The process for getting the top pipe dream is analogous, but with the labeling start from 1 along the top boundary, and a different order of reading (see fig below). Finally, in constructing the table as above which tells us which squares/transpositions are crosses in the pipe dream, we must use the inverse permutation. In our example the inverse permutation of 2225377 is 4255500.



Deletion and Contraction

Why consider deletion/contraction?

"Deletion and contraction of an element are fundamental operations in many areas...there is a tremendous power in proofs "by deletion and contraction" which proceed by induction on the number of elements, and by putting a structure together from the information given by deletion and contraction of the same element." (Lectures on Polytopes, Gunter Ziegler, 1995)

Indeed, we are not so much interested in deletion/contraction itself, but the thrust of this talk will be on using these operations to prove results about algebro-geometric properties of these varieties.

Deletion and Contraction of Matroids

Definition (Deletion of Matroids)

Restriction of a matroid $M = (E, B)$ to $S \subseteq E$ is denoted $M|_S$ with bases $B(M|_S) = \{b \cap S | b \in B \text{ and } |b \cap S| \text{ is maximal among all } b \in B\}$.
Deletion for $\tilde{S} \subseteq E$ is then restriction to $E \backslash \tilde{S}$.

Definition (Contraction of Matroids)

Contraction of this matroid by $T \subseteq E$ is denoted M/T with bases $B(M/T) = \{b - T | b \in B \text{ and } |b \cap T| \text{ is maximal among all } b \in B\}$

We care about deletion and contraction of a single element, that is, a single columns of the matrix.
For contraction, this means that for the intersection $b \cap T$ to be maximal, the base must contain the element T.
Fact: Deletion and contraction are dual operations, precisely: denoting by $*$ the dual operation of basis complement: $(M/S)^* = M^*|_{E-S}$.
Another fact: positroids are closed under deletion, contraction, and duals.

Deletion of Positroid Varieties

What about deletion and contraction for positroid varieties?
If we think about these in the matroid/positroid way, with nonzero determinants, a column let's say the ith column, is removed from every matroid if it is the zero column:

$$\begin{array}{c} i \\ \begin{pmatrix} & 0 & \\ & 0 & \\ & \vdots & \\ & 0 & \\ & 0 & \end{pmatrix} \end{array}$$

Deletion of Positroid Varieties

Contraction is only a little harder to see. We restrict to a set of matroids such that every basis contains this column.

When must a column be contained in every spanning set?
When it spans a subspace that no other column or combination of columns span.

So via a GL(k) transformation, it can be transformed to this form:

$$\begin{array}{c} \begin{matrix} & & i & & \end{matrix} \\ \begin{pmatrix} & & 0 & & \\ & & 0 & & \\ & & \vdots & & \\ & & 0 & & \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Deletion and Contraction of Positroid Varieties

Definition (Deletion/Contraction of Positroid Varieties)

Thus, for positroid varieties of Gr(k,n), deletion of the ith column of a positroid variety Π_f: the largest positroid variety $\Pi_{f'}$ contained in Π_f such that $f'(i) = 0$.

Contraction of the ith column of a positroid variety Π_f: the largest positroid variety $\Pi_{f'}$ contained in Π_f such that $f'(i) = n$.

Deletion and Contraction of Positroid Varieties (cont)

Proposition

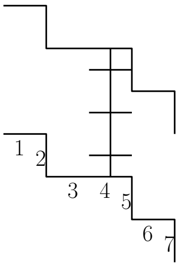
The deletion of the ith columns of a positroid variety is the same as taking the given positroid variety and intersecting it with the largest positroid variety with $f(i) = 0$, denoted $\Pi_{del,i}$.

$\Pi_{del,i}$ has siteswap $(k+1)^k 0 k^{n-k-1}$.
Analogously for contraction, with the largest positroid variety being $(k-1)^{n-k} k^{k-1} n$.

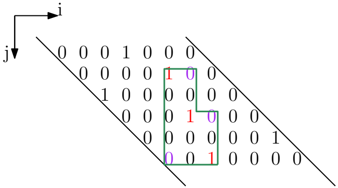
Proof: We are considering the intersection: $del_i(\Pi_f) = \Pi_f \cap \Pi_{del,i}$. Because the positroid varieties form a stratification, intersections of positroid varieties are unions of positroid varieties, but in this case, by the Białynicki-Birula Theorem, this intersection is irreducible (when it's nonempty), which means that $del(\Pi_f)$ cannot split into multiple positroid varieties - it must be a single positroid variety.

Deletion of Positroid Varieties with Pipe Dreams

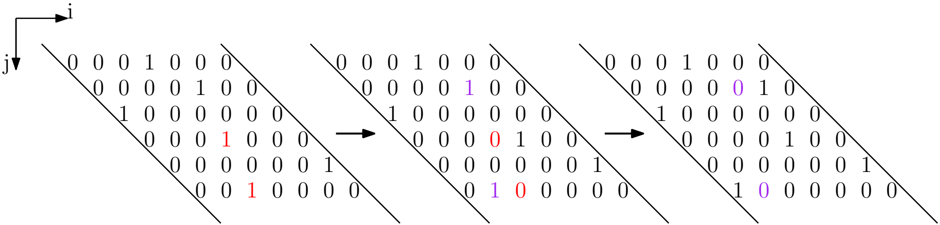
Here's what $\Pi_{del,i}$ for $i = 4$ looks like for the point $\{2, 5, 7\}$ in Gr(3,7):



It is perhaps easiest to say what is going on diagramatically:



Or, step by step:



Deletion of Siteswaps Algorithm

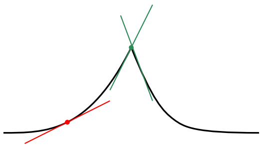
Here's the algorithm for the picture above.

Let g denote the bounded affine permutation.

Start with i and $g(i)$, we want $g(i) = i$ since this means $f(i) = 0$ Test $i-1$, $i-2$,... successively until the condition is reached. So we begin by testing $i-1$: if $i <= g(i-1) < g(i)$, then transpose i and $i-1$: define the new $g(i) := g(i-1)$ and $g(i-1) := g(i)$. So now $g(i)$ is a little closer to i. Continue until $g(i) = i$.

Smooth vs. Singular

We are interested in determining whether a positroid variety is singular or smooth. Intuitively, it is the notion that the tangent space is not "too big":



In this picture, the manifold drawn in black is 1-dimensional, and the red point is a smooth point since the tangent space is the right dimension, but the tangent space at the green point is spanned by the two lines, so it is too big.

Smooth Definitions

For completeness, we give the following definition of smoothness:
When the maximal ideal has number of generators equal to the dimension of the ring, that is, $dim I(a) = dim O_{X,a}$ So for a variety X, a part $a \in X$ is smooth if its ring of local function $A(X)_{I(a)}$ is regular, that is, if $dim T_a X = dim Hom_K(T_a X, K) = dim I(a)/I(a)^2 = codim_X\{a\} = dim X = dim A(X)_{I(a)}$. Equivalently, X is smooth at a if its tangent cone is the same as its tangent cone at a.
We often test smoothness using some sort of Jacobi condition, e.g.:

Theorem

Affine Jacobi Criterion
Let $a \in X$ be a point on an affine variety $X \subseteq \mathbb{A}^n$ and let $I(x) = \langle f_1, ..., f_n \rangle$. Then
X is smooth at $a \Leftrightarrow$ the rank of the $r \times n$ Jacobian matrix $(\frac{\partial f_i}{\partial x_j}(a))_{ij}$ is at least $n - codim_X\{a\}$ (so it's equal to $n - codim_X\{a\}$

Singular at Points

Theorem

We can check whether a positroid variety Π_f is singular (or smooth) just by testing singularity at the $\binom{n}{k}$ fixed points given by the span of k coordinate spaces.

Proof:
Fact (1): Borel's Fixed Point Theorem: If a solvable group (such as a torus) acts on some nonempty proper variety, then the fixed point set is also nonempty.
Fact (2): The torus T acts on the singular locus.
Proof: Whenever you have a group acting on a variety, then the singular locus will be G-invariant (i.e. a group of symmetries will leave the singular locus alone). The "proof" is simply that the notion of singularity does not require coordinates.
Fact (3): Positroid varieties are proper.(Projective ($\Rightarrow$) proper. Positroid varieties are closed in Grassmannians which are closed in projective space (in the Plucker embedding) so they're proper.)
Fact (4): The singular locus is closed (follows from smooth locus being given by a nonvanishing (open) determinantal Jacobian condition)

Singular at Points (cont)

Suppose Π_f has singular point(s), i.e. the singular locus $(\Pi_f)_{sing}$ is nonempty. Being closed in Π_f, it too is proper (using (3) and (4)). Thus, $(\Pi_f)_{sing}$ is proper, and by (2) the torus acts on it; therefore we can apply (1) Borel's Fixed point Theorem to it, to conclude that the singular locus includes at least one fixed point.

Denoting the fixed points of the torus action by superscript T, we have: $(\Pi_f)_{sing} \neq \emptyset \leftrightarrow (\Pi_f)^T_{sing} \neq \emptyset$

So we have reduced checking singularity (or smoothness) to a finite set of points $\{\lambda\} = \binom{[n]}{k}$!

In what follows, we will check these by checking on the affine open cover of the U_λ described earlier (referenced in a moment at affine pipe dreams. Clearly, this will include all fixed points $\{\lambda\}$.

Smooth Means One Pipe Dream

Theorem

Π_f is smooth at the point λ (it meets λ and is smooth there) if and only if there is a unique affine pipe dream on U_λ. So Π_f meets the point λ if there exists an affine pipe dream, and it is smooth if that affine pipe dream is unique. Thus, we can take Π_f and look at pipe dreams for all choices of λ to test for smoothness.

Proof idea: We have the multiplicity which tells us the degree of the tangent cone; this is 1 if and only if we are at a smooth point. The multiplicity is equal to the number of pipe dreams. This follows from looking at the T-equivariant cohomology class $[\Pi_f \cap U_\lambda]$ which can be computed by the AJS/Billey formula:

$$S_v|_w = \sum_{R \subseteq Q \text{ reduced}, P=Q, \Pi R = v\Pi_Q} (\alpha_q^{[q \in R]} r_q) \cdot 1$$

The important thing to note is that the rightmost terms can be set equal to one, in which case this formula is just computing a sum over reduced words for the permutation, which is just the number of pipe dreams.

Deletion of a Positroid Variety at a Point

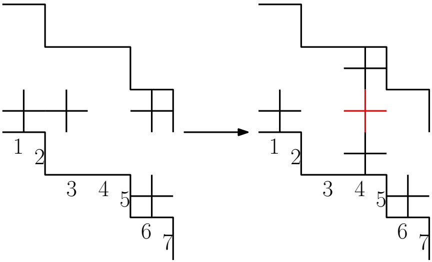
We can ask the question of what the deletion of a positroid variety Π_f at a given column i looks like on a patch U_λ. We already saw that this was given by the intersection: $del_i(\Pi_f) = \Pi_f \cap \Pi_{del,i}$, where $\Pi_{del,i}$ is the pipe dream with elbows everything except crosses along the column i.

So the question is: what does this intersection look like on U_λ?
Answer: it is given by filling in the pipe dream for Π_f on U_λ with crosses along the ith columns.

This prompts the immediate question: if there are multiple such pipe dreams for Π_f on U_λ, which one do we use?
Answer: we use those pipe dreams with the maximal number of crosses along column i.

Deletion of a Positroid Variety at a Point (cont)

For example:



That is, we move as many crosses to that columns as possible, and fill in the rest of the square with crosses (denoted by the red cross here).

Proof:
We take $del_i(\Pi_f) = \Pi_f \cap \Pi_{del,i}$ and intersect everything with the vector space U_λ:

$$del_i(\Pi_f) \cap U_\lambda = (\Pi_f \cap U_\lambda) \cap (\Pi_{del,i} \cap U_\lambda).$$

In other words, we get the components of $(\Pi_f \cap U_\lambda) \cap (\Pi_{del,i} \cap U_\lambda)$ by taking the components of the pipe dreams for Π_f and the components of the pipe dreams for $\Pi_{del,i}$ and intersecting them and looking at the maximal thing - since it the intersection is irreducible, there will be a maximal element that contains all others.

Furthermore, since we know that the maximal intersection will have the fewest number of crosses (since each cross is a variable that is set to zero, so an extra condition that lowers the dimension of the variety), we must choose to intersect with the pipe dream for $(\Pi_f \cap U_\lambda)$ with the greatest number of crosses along this column.

Questions (Repeated From Before)

(1) We know geometrically why singular things are closed under going up in deletion/contraction, while smooth things are closed under going down.

Theorem

If λ is smooth on Π_f, then $del_i(\lambda)$ is smooth on $\Pi_{del_i(f)}$.
Proof sketch: Put equivariant local coordinates near λ in which Π_f is a linear subspace.

Can we prove this in a combinatorial way?
(2) Can we describe the atomic positroid varieties?

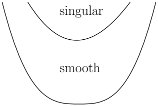
Definition (Atomic Positroid Variety)

A positroid variety is *atomic* if it is singular, but any deletion or contraction is takes it to a smooth (or empty) positroid variety.

(3) How can we quickly test whether a positroid variety is singular at a given point?

Question 1: Combinatorics of Smoothness Ordering

Question 1: Combinatorial proof that smooth positroid varieties remain smooth under deletion and contraction, while singular positroid varieties if obtained from another positroid variety via deletion or contraction must have come from a singular positroid variety:

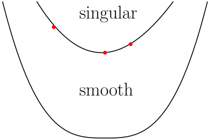


Proof: Deletion involves filling in a column with crosses. Contraction involves filling in a row with crosses. Neither can create a new move since it does not change the connectivity of the pipe dream.
Thus, if there were no moves prior to the deletion, then there are no moves after the deletion. This shows the converse as well, that a singular positroid variety must have come from a singular positroid variety, since a smooth positroid variety being deleted to a singular positroid variety is impossible.

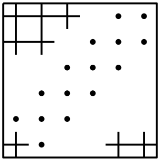
Question 2: Atomicity

Question 2: What do the atomic things look like?

In other words, what do the red dots in the figure below look like:



Answer: they look like (theorem, to be made more precise) the following:



Question 2: Atomicity (cont)

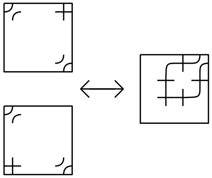
Specifically, they are square, so in $Gr(k, 2k)$ and on points where λ consists of k consecutive columns.

Proof:
The proof goes by showing that in order for any deletion or contraction to eliminate all moves, the crosses must be able to reach any column or row - this is only possible along the main diagonal. Everything else must be rigid (no moves), so they must be partitions in the northwest and southeast.

Note: any singular positroid variety at a point can be deleted/contracted until it reaches the atomic configuration, so being able to reach this atomic configuration under deletion and contraction is equivalent to being singular at that point. However, it is also easy to show this directly, by showing that any pipe dream with moves can be brought to this form.

Question 3: Singularity Determination at a Point

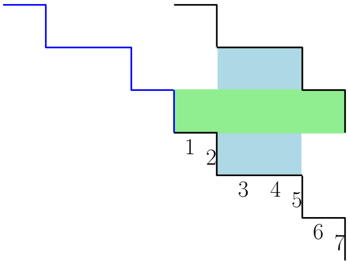
Question 3: How can we quickly determine whether a positroid variety is singular at a given point?
Answer a: we can compute the top and bottom pipe dreams as we described above and see if they match.
Answer b: globally, we can see if we can find a box with either of the two configuration on the left:



These two types of pattern are equivalent to there existing moves in the pipe dream, and I further showed that they are equivalent to seeing a "box" configuration like on the right of the figure.

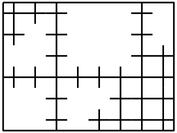
Question 3: Singularity Determination at a Point (cont)

Answer c: By our analysis of atomic things, we know that we can test to see whether a positroid variety is singular by seeing whether we can reach the atomic configuration - but all square configurations will be found in the maximal rectangles. This means that we just have to check smoothness/singularity on all maximal rectangles, for example we have drawn 2 overlapping one's in the figure below.



Question 3: Singularity Determination at a Point (cont)

In fact, by using the proof of answer b, we can show that the smooth rectangles are precisely of the form of deletion and contractions (rows and columns of crosses), plus partitions in the northwest and southeast, like this:



Main Theorem: Thus, if we check all maximal rectangles and see a configuation like this, then our positroid variety is smooth at the given point; otherwise, it is singular.

End

Thanks for listening!