

$\epsilon_{abcd} = \text{determinant}$

Phys 253B

D Problem (Schrodinger / Cramer's relation)

$$\lambda_3 = a\lambda_1 + b\lambda_2$$

$$|3\rangle = a|1\rangle + b|2\rangle$$

$$\langle 33 \rangle = a\langle 31 \rangle + b\langle 32 \rangle$$

$$\langle 23 \rangle = a\langle 21 \rangle + b\langle 22 \rangle$$

$$\langle 13 \rangle = a\langle 11 \rangle + b\langle 12 \rangle$$

$$\text{Then } a = \frac{\langle 23 \rangle}{\langle 21 \rangle} \quad b = \frac{\langle 13 \rangle}{\langle 11 \rangle}$$

$$\text{so } |3\rangle = \frac{\langle 23 \rangle}{\langle 21 \rangle} |1\rangle + \frac{\langle 13 \rangle}{\langle 11 \rangle} |2\rangle$$

hmm vectors?
 + don't need this via
 determinant: linearly independent so
 det(2,2)=0 also: 2x2
 determinant is antisymmetric

$$a|1\rangle + b|2\rangle + c|3\rangle = 0$$

$$a\langle 11 \rangle + b\langle 12 \rangle + c\langle 13 \rangle = 0$$

$$a\langle 21 \rangle + b\langle 22 \rangle + c\langle 23 \rangle = 0$$

$$a\langle 31 \rangle + b\langle 32 \rangle + c\langle 33 \rangle = 0$$

$$\begin{pmatrix} \langle 11 \rangle & \langle 12 \rangle & \langle 13 \rangle \\ \langle 21 \rangle & \langle 22 \rangle & \langle 23 \rangle \\ \langle 31 \rangle & \langle 32 \rangle & \langle 33 \rangle \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

Look for q.b. using Gaussian elimination
 Cramer's rule doesn't apply since the
 matrix is not invertible?

$$\langle 4 \rangle \quad \langle 21 \rangle |3\rangle - \langle 23 \rangle |1\rangle + \langle 13 \rangle |2\rangle = 0$$

$$\langle 21 \rangle \langle 43 \rangle - \langle 23 \rangle \langle 41 \rangle + \langle 13 \rangle \langle 42 \rangle = 0$$

$$\langle 42 \rangle \langle 34 \rangle + \langle 23 \rangle \langle 44 \rangle = \langle 13 \rangle \langle 24 \rangle$$

Generalize:

$$a_1 |1\rangle + a_2 |2\rangle + \dots + a_{k+1} |k+1\rangle = 0$$

$$|4\rangle = a|1\rangle + b|2\rangle + c|3\rangle + d|4\rangle = 0$$

$$\langle 14 \rangle = b\langle 12 \rangle + c\langle 13 \rangle$$

$$\langle 1+2 | 4 \rangle = a\langle 1+2 | 1 \rangle + b\langle 1+2 | 2 \rangle + c\langle 1+2 | 3 \rangle$$

$$a\langle 21 \rangle + b\langle 12 \rangle + c\langle 13 \rangle + c\langle 23 \rangle$$

$$(a-b)\langle 21 \rangle$$

$\langle \rangle$ are determinants?

p203 Lag: $A^{-1} = \frac{\text{adj } A}{\det A}$

$$a \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + b \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + c \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$aa_1 + bb_1 + cc_1 = 0$$

$$aa_2 + bb_2 + cc_2 = 0$$

$$\frac{a}{c} a_1 + \frac{b}{c} b_1 + c_1 = 0$$

$$\frac{a}{c} a_2 + \frac{b}{c} b_2 + c_2 = 0 \quad \cdot -\frac{a_1}{a_2}$$

$$-\frac{b_1}{b_2} \cdot \checkmark$$

$$-\frac{a}{c} a_1 - \frac{b}{c} \frac{b_2 a_1}{a_2} - \frac{c_2 a_1}{a_2} = 0$$

$$\frac{b}{c} \left(b_1 - \frac{b_2 a_1}{a_2} \right) + c_1 - \frac{c_2 a_1}{a_2} = 0$$

$$\begin{bmatrix} \frac{b}{c} \\ c \end{bmatrix} = \frac{\frac{c_2 a_1}{a_2} - c_1}{b_1 - \frac{b_2 a_1}{a_2}} = \frac{\frac{c_2 a_1 - c_1 a_2}{a_2}}{\frac{b_1 a_2 - b_2 a_1}{a_2}} = \frac{c_2 a_1 - c_1 a_2}{b_1 a_2 - b_2 a_1} = \frac{\det(ac)}{\det(ba)}$$

$$-\frac{b_1 a}{b_2 c} a_2 + \frac{b}{c} b_1 + \frac{c_2 b_1}{b_2}$$

$$\frac{a}{c} \left(a_1 - \frac{b_1 a_2}{b_2} \right) + c_1 - \frac{c_2 b_1}{b_2} = 0$$

$$\boxed{\frac{a}{c}} = \frac{\frac{c_2 b_1}{b_2} - c_1}{a_1 - \frac{b_1 a_2}{b_2}}$$

$$= \frac{c_2 b_1 - c_1 b_2}{b_2 a_1 - b_1 a_2}$$

$$= \frac{c_2 b_1 - c_1 b_2}{b_2 a_1 - b_1 a_2}$$

$$= \frac{\langle bc \rangle}{\langle ab \rangle}$$

$$\rightarrow \frac{a}{c} a_1 + \frac{\langle ac \rangle}{\langle ba \rangle} b_1 + c_1 = 0$$

$$\frac{a}{c} a_1 = -\frac{\langle ac \rangle}{\langle ba \rangle} b_1 - c_1 \Rightarrow \frac{a}{c} = -\frac{\langle ac \rangle}{\langle ba \rangle} \frac{b_1}{a_1} - \frac{c_1}{a_1}$$

$$\text{By} \quad \textcircled{3} \quad \frac{a}{c} = -\frac{\langle ac \rangle}{\langle ba \rangle} \frac{b_2}{a_2} - \frac{c_2}{a_2}$$

$$\frac{\langle bc \rangle}{\langle ab \rangle} |a\rangle + \frac{\langle ac \rangle}{\langle ba \rangle} |b\rangle + |c\rangle = 0$$

$$\langle 23 | 1 \rangle - \langle 13 | 2 \rangle + \langle 12 | 3 \rangle = 0$$

Use induction:

Remarks re § 1.1.1 p20 Cor. Little's Lemma

Suppose for k vectors in $(k-1)$ -dim:

$$\langle v_2 \dots v_k \rangle v_1 - \langle v_1 v_3 \dots v_k \rangle v_2 + \langle v_1 v_2 v_4 \dots v_k \rangle v_3 + \dots$$

are these the
coefficients of?

$$+ \dots + (-1)^{k+1} \langle v_1 v_2 \dots v_{k-1} \rangle v_k = 0$$

$$\text{WTS: } \langle v_2 \dots v_{k+1} \rangle v_1 - \langle v_1 v_3 \dots v_{k+1} \rangle v_2 + \dots + (-1)^{k+2} \langle v_1 v_2 \dots v_k \rangle v_{k+1} = 0$$

$$\begin{vmatrix} 0 & 0 & 0 & \dots & 0 \\ v_1 & \dots & v_{k+1} \end{vmatrix} \rightarrow \text{determinant zero}$$

$$\text{by: } \begin{vmatrix} 1 & 1 & \dots & 1 \\ v_1 & \dots & v_{k+1} \end{vmatrix} \xrightarrow{\text{det}} = \langle v_2 \dots v_{k+1} \rangle - \langle v_1 \dots v_{k+1} \rangle + \dots$$

$$\text{or: } Ax = b: A_x = \begin{vmatrix} 0 & 0 & \dots & 0 \\ v_1 & \dots & v_{k+1} \end{vmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{Then } x_1 = \frac{\langle v_1 \dots \hat{v}_1 \dots v_{k+1} \rangle (-1)^n}{\det A} = 0$$

$$a|a\rangle + b|b\rangle + c|c\rangle + d|d\rangle = 0$$

$$a \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + b \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + c \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + d \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = 0$$

$$\begin{aligned} a a_1 + b b_1 + c c_1 + d d_1 &= 0 \\ a a_2 + b b_2 + c c_2 + d d_2 &= 0 \\ a a_3 + b b_3 + c c_3 + d d_3 &= 0 \end{aligned} \Rightarrow \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = - \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\text{Cramer's rule: } x_i = \frac{\det A_i(b)}{\det A}$$

$$\begin{aligned} \text{or } \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= -A^{-1} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \\ &= -\frac{\text{adj} A}{\det A} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \\ &= \frac{\langle 1 \dots \overset{k+1}{\cancel{k}} \dots k \rangle}{\langle 1 2 \dots k \rangle} = \frac{(-1)^{k-i} \langle 1 \dots i \dots k k+1 \rangle}{\langle 1 2 \dots k \rangle} \end{aligned}$$

$$\text{so } \lambda_1 |a_1\rangle + \lambda_2 |a_2\rangle + \dots + \lambda_{k+1} |a_{k+1}\rangle = 0$$

$$= \frac{\langle 2 \dots k+1 \rangle}{\langle 1 2 \dots k \rangle} |a_1\rangle - \frac{\langle 1 3 \dots k+1 \rangle}{\langle 1 2 \dots k \rangle} |a_2\rangle + \dots + |a_{k+1}\rangle = 0$$

$$\Rightarrow \langle 2 \dots k+1 \rangle |a_1\rangle - \langle 1 3 \dots k+1 \rangle |a_2\rangle + \dots + |a_{k+1}\rangle = 0$$

3) Functio Paritatis facta

$$P(1, 2, \dots, n) = \frac{1}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

$$P(1, 2, 3, 4) + P(1, 3, 2, 4) + P(1, 3, 4, 2) = 0$$

$$\frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} + \frac{1}{\langle 13 \rangle \langle 32 \rangle \langle 24 \rangle \langle 41 \rangle} + \frac{1}{\langle 13 \rangle \langle 34 \rangle \langle 42 \rangle \langle 21 \rangle}$$

$$= \frac{\langle 24 \rangle \langle 13 \rangle + \langle 12 \rangle \langle 34 \rangle + \langle 23 \rangle \langle 41 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \langle 24 \rangle \langle 13 \rangle}$$

$$= \frac{\langle 41 \rangle (-\langle 13 \rangle \langle 24 \rangle + \langle 12 \rangle \langle 34 \rangle + \langle 23 \rangle \langle 14 \rangle)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle \langle 24 \rangle \langle 13 \rangle} = \frac{\langle 41 \rangle (0)}{\langle 12 \rangle \dots \langle 13 \rangle} = 0$$

by Schur's problem D? but this Schur's is only for 2-D vectors?

$$P(1, 2, \dots, n) + P(1, 3, 2, \dots, n) + \dots + P(1, 3, 4, \dots, 2) = 0$$

$$\frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \dots \langle n1 \rangle} + \frac{1}{\langle 13 \rangle \langle 32 \rangle \langle 24 \rangle \dots \langle n1 \rangle} + \frac{1}{\langle 13 \rangle \langle 34 \rangle \langle 42 \rangle \langle 25 \rangle \dots \langle n1 \rangle}$$

$$+ \dots + \frac{1}{\langle 13 \rangle \langle 34 \rangle \langle 45 \rangle \dots \langle n-1, 2 \rangle \langle 2n \rangle \langle n1 \rangle} + \frac{1}{\langle 13 \rangle \langle 34 \rangle \dots \langle n-1, n \rangle \langle n2 \rangle \langle 21 \rangle}$$

$$= \langle 24 \rangle \langle 25 \rangle \dots \langle 2n \rangle \pm \langle 12 \rangle \langle 34 \rangle \langle 25 \rangle \dots \langle 2n \rangle \pm \dots \pm \langle n1 \rangle \langle 23 \rangle \dots \langle 2n-1 \rangle$$

$$= \langle 13 \rangle \langle 24 \rangle \langle 25 \rangle \dots \langle 2n \rangle + (-) \langle 34 \rangle \langle 12 \rangle \langle 25 \rangle \dots \langle 2n \rangle + (-) \langle 45 \rangle \langle 12 \rangle \langle 23 \rangle \dots \langle 2n \rangle + \dots$$

$$+ \langle 43 \rangle \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \dots \langle n1 \rangle \cdot \langle 24 \rangle \langle 25 \rangle \dots \langle 2n \rangle$$

$$= -\langle 13 \rangle \langle 24 \rangle \langle 25 \rangle \dots \langle 2n \rangle + \langle 34 \rangle \langle 21 \rangle \langle 25 \rangle \dots \langle 2n \rangle + \langle 45 \rangle \langle 21 \rangle \langle 23 \rangle \langle 26 \rangle \dots \langle 2n \rangle$$

$$+ \dots + \langle 11 \rangle \langle 21 \rangle \langle 23 \rangle \dots \langle 2n-1 \rangle$$

$$\langle 13 \rangle \langle 34 \rangle \langle 45 \rangle \dots \langle n1 \rangle \langle 21 \rangle \langle 23 \rangle \langle 24 \rangle \dots \langle 2n \rangle$$

WTS this is equal to zero

$$\Rightarrow \langle 13 \rangle \langle 24 \rangle \dots \langle 2n \rangle + \langle 21 \rangle \langle 34 \rangle \langle 25 \rangle \dots \langle 2n \rangle + \langle 21 \rangle \langle 23 \rangle \langle 45 \rangle \langle 26 \rangle \dots \langle 2n \rangle + \dots + \langle 21 \rangle \langle 23 \rangle \dots \langle 2n-1 \rangle \langle n1 \rangle$$

$$\underbrace{\langle 11 \rangle \langle 23 \rangle \dots \langle 2n \rangle}_{\langle 13 \rangle}$$

$$\begin{pmatrix} \langle 21 \rangle \langle 21 \rangle \\ \langle 23 \rangle \langle 23 \rangle \\ \vdots \\ \langle 2n \rangle \end{pmatrix}$$

$$\langle i2 \rangle \langle 2j \rangle$$

$$2 \times 2 \quad \lambda_2^\alpha \lambda_{2\alpha}$$

$$= -\frac{\langle 22 \rangle}{n-1} \left(\langle 13 \rangle \langle 24 \rangle \dots \langle 2n \rangle + \dots + \langle 23 \rangle \dots \langle 2n-1 \rangle \langle n1 \rangle \right) \quad \text{if } \langle i2 \rangle \langle 2j \rangle = -\langle j2 \rangle \langle 2i \rangle$$

Is this zero?

What was the formula for $\begin{pmatrix} \langle i1 \rangle \\ \vdots \end{pmatrix}$ as determinant

$$a_1 dx_1 \wedge \dots \wedge a_n dx_n = \det(a_1 \dots a_n) dx_1 \wedge \dots \wedge dx_n$$

$$(\langle 11 \rangle + \langle 21 \rangle + \dots + \langle n1 \rangle) \wedge (\langle 11 \rangle + \langle 12 \rangle + \dots + \langle 1n \rangle)$$

⊗?

$$\det \begin{pmatrix} \langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle \\ \langle 23 \rangle \dots \langle n1 \rangle \langle 12 \rangle \end{pmatrix} = (\langle 11 \rangle + \langle 21 \rangle + \dots + \langle n1 \rangle) \wedge (\langle 11 \rangle + \langle 12 \rangle + \dots + \langle 1n \rangle) \wedge \dots \wedge (\langle 11 \rangle + \langle 12 \rangle + \dots + \langle 1n \rangle)$$

$$\begin{pmatrix} \langle 13 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \\ \langle 21 \rangle \langle 24 \rangle \langle 21 \rangle \langle 21 \rangle \langle 21 \rangle \\ \langle 23 \rangle \langle 23 \rangle \langle 25 \rangle \langle 25 \rangle \langle 23 \rangle \\ \langle 26 \rangle \langle 24 \rangle \langle 24 \rangle \langle 26 \rangle \langle 26 \rangle \\ \langle 27 \rangle \langle 27 \rangle \langle 27 \rangle \langle 25 \rangle \langle 27 \rangle \end{pmatrix}$$

$$\langle 13 \rangle \langle 24 \rangle = \langle 34 \rangle \langle 21 \rangle$$

$$\langle 13 \rangle \langle 24 \rangle \langle 25 \rangle = \langle 13 \rangle \langle 21 \rangle \langle 23 \rangle + \langle 34 \rangle \langle 21 \rangle \langle 23 \rangle$$

$$- \langle 34 \rangle \langle 21 \rangle \langle 25 \rangle + \langle 45 \rangle \langle 21 \rangle \langle 23 \rangle - \langle 45 \rangle \langle 24 \rangle \langle 23 \rangle$$

$$\lambda_a = t_a \times C_1, z_a$$

$$\lambda_a = t_a \begin{pmatrix} 1 \\ z_a \end{pmatrix} \quad \text{so} \quad \lambda_a = \begin{pmatrix} \lambda_a^1 \\ \lambda_a^2 \end{pmatrix} = \lambda_a^1 \begin{pmatrix} 1 \\ \lambda_a^2 / \lambda_a^1 \end{pmatrix}$$

$$\text{Schur's: } \langle 13 \rangle \langle 24 \rangle = \langle 12 \rangle \langle 34 \rangle + \langle 23 \rangle \langle 14 \rangle$$

$$\text{or } \langle 23 \rangle \lambda_1 - \langle 13 \rangle \lambda_2 + \langle 12 \rangle \lambda_3 = 0$$

$$\langle 23 \rangle = \det t_2 t_3 \begin{pmatrix} 1 & 1 \\ z_2 & z_3 \end{pmatrix} = t_2 t_3 (z_3 - z_2)$$

$$t_1 t_2 t_3 (z_3 - z_2) \begin{pmatrix} 1 \\ z_1 \end{pmatrix} - t_1 t_2 t_3 (z_3 - z_1) \begin{pmatrix} 1 \\ z_2 \end{pmatrix} + t_1 t_2 t_3 (z_2 - z_1) \begin{pmatrix} 1 \\ z_3 \end{pmatrix}$$

$$= t_1 t_2 t_3 \begin{pmatrix} z_3 - z_2 - (z_3 - z_1) + z_2 - z_1 \\ z_1 z_3 - z_1 z_2 - (z_2 z_3 - z_1 z_2) + z_3 (z_2 - z_1) \end{pmatrix} = t_1 t_2 t_3 \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$PT(1, 2, \dots, n) = \frac{1}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} = \frac{1}{t_1 t_2 (z_2 - z_1) \cdot t_2 t_3 (z_3 - z_2) \dots t_n t_1 (z_1 - z_n)}$$

$$= \frac{1}{t_1^2 t_2^2 \dots t_n^2 (z_2 - z_1)(z_3 - z_2) \dots (z_1 - z_n)}$$

$$\text{Mag WTS: } PT(1, 2, \dots, n) + PT(1, 3, 2, \dots, n) + \dots + PT(1, 3, 4, \dots, 2) = 0$$

$$= \frac{1}{t_1^2 t_2^2 \dots t_n^2} \left(\frac{1}{(z_2 - z_1)(z_3 - z_2) \dots (z_n - z_1)} + \frac{1}{(z_3 - z_1)(z_2 - z_3) \dots (z_n - z_1)} + \dots + \frac{1}{(z_3 - z_1)(z_4 - z_3) \dots (z_2 - z_n)} \right)$$

$$\text{suppl} \rightarrow = \frac{1}{n!} \left(\underbrace{-(z_3 - z_1)(z_4 - z_3)(z_5 - z_4) \dots (z_n - z_2) + (z_4 - z_3)(z_5 - z_4) \dots (z_n - z_2) + (z_1 - z_n)(z_1 - z_2) \dots (z_{n-1} - z_2)}_{=0} \right)$$

Use induction: Base case: prove

$$\text{Now: suppose } -\langle 13 \rangle \langle 24 \rangle \langle 25 \rangle + \dots + \langle 2n-1 \rangle + \langle 34 \rangle \dots \langle 2n-1 \rangle + \langle n-1 \rangle \langle 2n \rangle + \langle 2n \rangle = 0$$

Base case: $PT(1, 2, 3) + PT(1, 3, 2) = 0$

$$\frac{1}{\langle 12 \rangle \langle 23 \rangle} + \frac{1}{\langle 13 \rangle \langle 32 \rangle} = \frac{\langle 13 \rangle - \langle 12 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 13 \rangle} = 0$$

$$PT(1, 2, 3, 4) + PT(1, 3, 2, 4) + PT(1, 3, 4, 2)$$

$$= \langle 13 \rangle \langle 24 \rangle - \langle 12 \rangle \langle 34 \rangle + \langle 23 \rangle \langle 41 \rangle$$

$$= \langle 13 \rangle \langle 24 \rangle + \langle 34 \rangle \langle 21 \rangle + \langle 41 \rangle \langle 23 \rangle$$

$$PT(1, 2, 3, 4, 5) + PT(1, 3, 2, 4, 5) + PT(1, 3, 4, 2, 5) + PT(1, 3, 4, 5, 2)$$

$$= \langle 13 \rangle \langle 24 \rangle \langle 25 \rangle + \langle 34 \rangle \langle 21 \rangle \langle 25 \rangle + \langle 45 \rangle \langle 21 \rangle \langle 23 \rangle + \langle 51 \rangle \langle 23 \rangle \langle 24 \rangle$$

If we multiply by $\langle 25 \rangle \rightarrow$ last term: $\langle 41 \rangle \langle 23 \rangle \langle 25 \rangle$

$$n-1=4: \quad \langle n-1, 1 \rangle \langle 23 \rangle \langle 24 \rangle \dots \langle 2n-2 \rangle \langle 2n \rangle$$

||?

$$\langle n-1, n \rangle \langle 21 \rangle \langle 23 \rangle \dots \langle 2n-2 \rangle + \langle n, 1 \rangle \langle 23 \rangle \langle 24 \rangle \dots \langle 2n-1 \rangle$$

$$\Rightarrow \text{WTS: } \langle n-1, 1 \rangle \langle 2n \rangle = \langle n-1, n \rangle \langle 21 \rangle + \langle n, 1 \rangle \langle 2n-1 \rangle$$

$$(x_{n-1} - x_1)(x_n - x_2) = (x_{n-1} - x_n)(x_2 - x_1) + (x_n - x_1)(x_2 - x_{n-1})$$

$$(x_1 - x_{n-1})(x_n - x_2)$$

$$(x_n - x_{n-1})(x_1 - x_2) + (x_1 - x_n)(x_{n-1} - x_2)$$

$$x_1 x_n - x_1 x_2 - x_{n-1} x_n + x_2 x_{n-1}$$

$$x_1 x_n - x_2 x_n - x_1 x_{n-1} + x_2 x_{n-1} + (x_1 x_{n-1} - x_1 x_2 - x_{n-1} x_n + x_2 x_n)$$

$$\rightarrow = x_1 x_n - x_1 x_2 + x_2 x_{n-1} - x_{n-1} x_n$$

$$f_{\mu\nu} = p_\mu \epsilon_\nu - p_\nu \epsilon_\mu \text{ is gauge invariant}$$

$$= \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \epsilon_{\beta\dot{\beta}} - \lambda^\beta \tilde{\lambda}^{\dot{\beta}} \epsilon_{\alpha\dot{\alpha}}$$

↓ Off diagonal matter

$$= \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} (\epsilon_{\beta\dot{\beta}} + k_1 \lambda^\beta \tilde{\lambda}^{\dot{\beta}}) - \lambda^\beta \tilde{\lambda}^{\dot{\beta}} (\epsilon_{\alpha\dot{\alpha}} + k_2 \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}})$$

$$= \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \epsilon_{\beta\dot{\beta}} - \lambda^\beta \tilde{\lambda}^{\dot{\beta}} \epsilon_{\alpha\dot{\alpha}} + k_1 \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \lambda^\beta \tilde{\lambda}^{\dot{\beta}} - k_2 \lambda^\beta \tilde{\lambda}^{\dot{\beta}} \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$$

there cancel if you shift both auxiliary spinors from ϵ_ν and ϵ_μ are the same as you shift by the same thing

$$f_{\alpha\beta\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} \lambda_\alpha \lambda_\beta \text{ is manifestly independent of } \tilde{\mu}$$

$$\epsilon_{\alpha\dot{\alpha}} = \frac{\tilde{\mu}_\alpha \lambda_{\dot{\alpha}}}{[\tilde{\mu}\lambda]}, \quad \epsilon_{\beta\dot{\beta}} = \frac{\tilde{\mu}_\beta \lambda_{\dot{\beta}}}{[\tilde{\mu}\lambda]}, \quad \epsilon_{\dot{\alpha}\dot{\beta}} = \frac{\tilde{\mu}_{\dot{\beta}} \tilde{\lambda}_{\dot{\alpha}}}{[\tilde{\mu}\lambda]}$$

$$f_{\alpha\beta\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} \lambda_\alpha \lambda_\beta = \frac{\tilde{\mu}_{\dot{\beta}} \tilde{\lambda}_{\dot{\alpha}} \lambda_\alpha \lambda_\beta}{[\tilde{\mu}\lambda]}$$

$$\lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \frac{\tilde{\mu}_{\dot{\beta}} \lambda_\beta}{[\tilde{\mu}\lambda]} - \lambda^\beta \tilde{\lambda}^{\dot{\beta}} \frac{\tilde{\mu}_{\dot{\alpha}} \lambda_\alpha}{[\tilde{\mu}\lambda]} = \frac{\lambda_\alpha \lambda_\beta (\tilde{\lambda}^{\dot{\alpha}} \tilde{\mu}_{\dot{\beta}} - \tilde{\lambda}^{\dot{\beta}} \tilde{\mu}_{\dot{\alpha}})}{[\tilde{\mu}\lambda]}$$

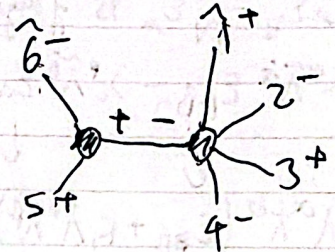
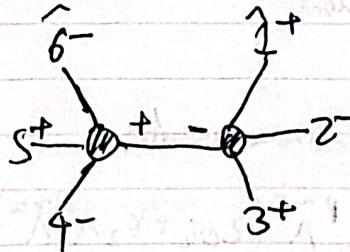
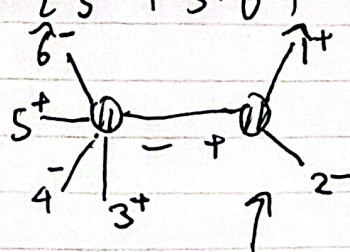
$$\left. \begin{aligned} \dot{\alpha}=0, \dot{\beta}=1: & \frac{\tilde{\lambda}^0 \tilde{\mu}_1 - \tilde{\lambda}^1 \tilde{\mu}_0}{\tilde{\mu}^0 \tilde{\lambda}_0 - \tilde{\mu}^1 \tilde{\lambda}_1} = 1 \\ \dot{\alpha}=1, \dot{\beta}=0: & = -1 \\ \dot{\alpha}=\dot{\beta}=0, \dot{\alpha}=\dot{\beta}=1: & = 0 \end{aligned} \right\} \epsilon_{\dot{\alpha}\dot{\beta}}$$

$$\frac{1}{[\tilde{\mu}\lambda] + [\tilde{\nu}\lambda]} = \frac{1}{[\tilde{\mu}\lambda]} \frac{1}{1 + \frac{[\tilde{\nu}\lambda]}{[\tilde{\mu}\lambda]}} \sim \frac{1}{[\tilde{\mu}\lambda]} \left(1 - \frac{[\tilde{\nu}\lambda]}{[\tilde{\mu}\lambda]} \right)$$

$$\text{so } \frac{\tilde{\mu}_\lambda + \tilde{\nu}_\lambda}{[\tilde{\mu}\lambda] + [\tilde{\nu}\lambda]} = \frac{\tilde{\mu}_\lambda + \tilde{\nu}_\lambda}{[\tilde{\mu}\lambda]} \left(1 - \frac{[\tilde{\nu}\lambda]}{[\tilde{\mu}\lambda]} \right) = \frac{\tilde{\mu}_\lambda + \tilde{\nu}_\lambda}{[\tilde{\mu}\lambda]} - \frac{(\tilde{\mu}_\lambda + \tilde{\nu}_\lambda) [\tilde{\nu}\lambda]}{[\tilde{\mu}\lambda]^2} = \frac{\lambda (\tilde{\mu} [\tilde{\mu}\lambda] + \tilde{\nu} [\tilde{\mu}\lambda] - \tilde{\mu} [\tilde{\nu}\lambda] - \tilde{\nu} [\tilde{\mu}\lambda])}{[\tilde{\mu}\lambda]^2}$$

$$\begin{aligned} & \text{The thing in the numerator is proportional to } \tilde{\lambda} \text{ since } \tilde{\lambda} (\tilde{\mu} [\tilde{\mu}\lambda] + \tilde{\nu} [\tilde{\mu}\lambda] - \tilde{\mu} [\tilde{\nu}\lambda] - \tilde{\nu} [\tilde{\mu}\lambda]) \\ & = [\tilde{\lambda}\tilde{\mu}] [\tilde{\mu}\lambda] + [\tilde{\lambda}\tilde{\nu}] [\tilde{\mu}\lambda] - [\tilde{\mu}\tilde{\lambda}] [\tilde{\nu}\lambda] - [\tilde{\nu}\tilde{\lambda}] [\tilde{\mu}\lambda] = 0 \end{aligned}$$

Problem 6: Six-point gluon scatter
 $(1^+ 2^- 3^+ 4^- 5^+ 6^-)$



~~If I~~ the shift is happen on 1^+ ,

$$\text{so } \hat{\lambda}_{\alpha}^{(1)} + z \hat{\lambda}_{\alpha}^{(6)}$$

$$D \rightarrow D + z [6]$$

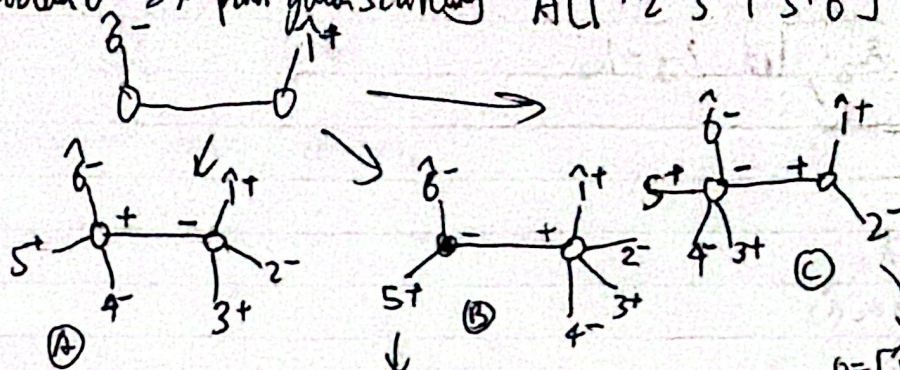
Must be $++-$ $\rightarrow \lambda$'s parallel, depends only on λ
 otherwise the shift breaks the parallel

Can we compute the one in the middle using 2 Parke-Taylor?

$$\frac{\langle 64 \rangle^4}{\langle 14 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle} \cdot \frac{1}{P_1^2} \frac{\langle 21 \rangle^4}{\langle 11 \rangle \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle}$$

$$P_1^2 = (p_1 + p_2 + p_3)^2$$

Problem 6: Six-point gluon scattering $A[1^+ 2^- 3^+ 4^- 5^+ 6^-]$



$$[6, 1] = [-+] \\ \hat{6} = \hat{6} + \epsilon \hat{1} \\ \hat{1} = \hat{1} - \epsilon \hat{6}$$

$$(\mu 56 \epsilon \mu) \quad [\hat{6} 5] \langle \hat{6} 5 \rangle = 0 \Rightarrow [\hat{6} 5] = 0 \\ [\hat{6} 5] \langle \hat{6} 5 \rangle \quad \text{so must have } \epsilon \text{ from } \langle 56 \rangle \text{ dependence}$$

$$0 = [\hat{1} 2] \langle \hat{1} 2 \rangle = [12] \langle 12 \rangle \Rightarrow \langle 12 \rangle = 0 \\ \Rightarrow \text{must depend on } [12] \quad \epsilon \hat{1}$$

Needed (C) on p 111-112

$$\frac{[\hat{6} \hat{1}_2] [\hat{1}_2 3] [34] [45] [56]}{[\hat{6} \hat{1}_2] [\hat{1}_2 3] [34] [45] [56]} \cdot \frac{1}{[12] \langle 12 \rangle} \cdot \frac{[\hat{1} \hat{1}_2]^3}{[12] [\hat{1}_2 2]}$$

$$0 = \hat{1}_2^2 = \langle 12 \rangle [\hat{1} 2] = \langle 12 \rangle [12] = (1 - \epsilon \langle 6 \rangle) [12] = \langle 12 \rangle - \epsilon \langle 62 \rangle [12] = 0$$

$$\hat{1}_2 = [\hat{1} + p_2] = \hat{1} + p_2 = \hat{1} + 2 = [1+2]$$

$$\hat{1}_2 = \hat{1} + p_2 = \hat{1} + 2 = (1 - \epsilon \langle 6 \rangle) [1+2] = 1[1+2] - \epsilon \langle 6 \rangle [1+2]$$

$$= \hat{1} + p_2 - \epsilon \langle 6 \rangle [1+2] = \hat{1} + p_2 - \frac{42}{62} \epsilon \langle 6 \rangle [1+2] \quad ? \text{ Needed } \hat{1}_2 = \frac{(\hat{1} + p_2) [16]}{\langle 26 \rangle} \quad ? \text{ p 110} \xrightarrow{\text{by direct proof}}$$

$$\text{Using EH rule: } \frac{[\hat{1} \hat{1}_2]^3}{[\hat{6} \hat{1}_2] [\hat{1}_2 3] [\hat{1}_2 2]} \cdot \frac{\langle \hat{1}_2 6 \rangle^3}{\langle \hat{1}_2 6 \rangle} = \frac{[12]^3 \langle 26 \rangle^3}{[12] \langle 16 \rangle ([3] \langle 16 \rangle + [23] \langle 26 \rangle) ([6] \langle 16 \rangle + [26] \langle 26 \rangle)}$$

$$[\hat{1} \hat{1}_2] \langle \hat{1}_2 6 \rangle = [12] \langle 26 \rangle$$

$$[3 \hat{1}_2] \langle \hat{1}_2 6 \rangle = [31] \langle 16 \rangle + [32] \langle 26 \rangle$$

$$[2 \hat{1}_2] \langle \hat{1}_2 6 \rangle = [21] \langle 16 \rangle$$

$$[\hat{6} \hat{1}_2] \langle \hat{1}_2 6 \rangle = -2 \hat{p}_6 \cdot \hat{1}_2 = -2 \hat{p}_6 \cdot (p_3 + p_4 + p_5 + \hat{p}_6) = -2 \hat{p}_6 (p_3 + p_4 + p_5) = -2 (\hat{p}_6 p_3 + \hat{p}_6 p_4 + \hat{p}_6 p_5)$$

$$= (p_3 + p_4 + p_5 + \hat{p}_6)^2 - (p_3 + p_4 + p_5)^2 = \hat{1}_2^2 - (p_3 + p_4 + p_5)^2 = -(p_3 + p_4 + p_5)^2$$

$$= 2(p_6 p_3 + p_6 p_4 + p_6 p_5 + p_3 p_4 + p_3 p_5 + p_4 p_5)$$

$$= [\hat{6} \hat{1}] \langle \hat{1} 6 \rangle + [\hat{6} 2] \langle 26 \rangle$$

$$= [\hat{6} 1] \langle 16 \rangle + [62] \langle 62 \rangle$$

$$\hat{1}_6^2$$

$$\hat{1}_6^2$$

Now: let, for $\lambda_I \propto \lambda_1 \propto \lambda_2$, let $\lambda_I = \lambda_2$

solve for $\tilde{\lambda}_I$ by method casimir:

$$\lambda_I \tilde{\lambda}_I = \lambda_1 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2$$

$$\lambda_2 \tilde{\lambda}_I = (\lambda_1 + \epsilon \lambda_6) \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2$$

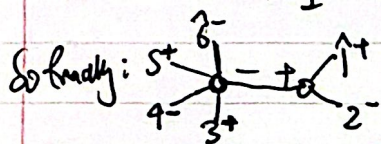
$$= \lambda_1 \tilde{\lambda}_1 + \frac{\langle 42 \rangle}{\langle 62 \rangle} \lambda_6 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2$$

$$= \frac{(\langle 62 \rangle \lambda_1 + \langle 12 \rangle \lambda_6)}{\langle 62 \rangle} \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2$$

Schouten

$$= \frac{\langle 61 \rangle \lambda_2 \tilde{\lambda}_1}{\langle 62 \rangle} + \lambda_2 \tilde{\lambda}_2$$

$$\Rightarrow \tilde{\lambda}_I = \frac{\langle 61 \rangle}{\langle 62 \rangle} \tilde{\lambda}_1 + \tilde{\lambda}_2 = \frac{\langle 61 \rangle \tilde{\lambda}_1 + \langle 62 \rangle \tilde{\lambda}_2}{\langle 62 \rangle} = \frac{(p_1 + p_2) 16}{\langle 62 \rangle}$$



$$= \frac{[35]^2}{[13][34][45][56][61]}$$

$$\left\{ \frac{1}{[12][12]} \cdot \frac{[11]^3}{[12][12]} \right\}$$

Now: take $\tilde{\lambda}_I - \frac{\langle 61 \rangle}{\langle 62 \rangle} \tilde{\lambda}_1 - \tilde{\lambda}_2 = 0$ cast it over:

$$1: [11] = [21]$$

$$2: [12] = \frac{\langle 61 \rangle}{\langle 62 \rangle} [12]$$

$$\frac{1}{[12]^2 \langle 12 \rangle} \cdot \frac{1}{\frac{\langle 61 \rangle}{\langle 62 \rangle} [12]} = \frac{\langle 62 \rangle}{\langle 61 \rangle \langle 12 \rangle} \rightarrow \frac{1}{[12] \langle 12 \rangle} \cdot \frac{[12]^3}{\frac{\langle 61 \rangle}{\langle 62 \rangle} [12] [12]} = -\frac{\langle 62 \rangle}{\langle 12 \rangle \langle 61 \rangle}$$

$$= \frac{[35]^2}{[34][45][56][61][13]} \cdot \frac{\langle 62 \rangle}{\langle 61 \rangle \langle 12 \rangle}$$

$$\frac{[51 p_6 + p_1 12]}{\langle 62 \rangle} \cdot \frac{\langle 2(p_1 + p_6)(p_1 + p_2) 16 \rangle}{\langle 62 \rangle \langle 26 \rangle} \cdot \frac{\langle 6(p_1 + p_2) 3 \rangle}{\langle 26 \rangle}$$

$$\frac{1}{\langle 62 \rangle^4} ([56] \langle 62 \rangle + [51] \langle 12 \rangle) \cdot (2(p_1^2 + p_1 p_2 + p_6 p_1 + p_6 p_2) 16) \cdot (\langle 61 \rangle [13] + \langle 62 \rangle [23])$$

$$\langle 21 \rangle [12] \langle 26 \rangle + \langle 26 \rangle [61] \langle 16 \rangle + \langle 26 \rangle [62] \langle 26 \rangle$$

$$= \langle 26 \rangle (p_1 p_2 + p_1 p_6 + p_2 p_6) (-1)$$

$$= \langle 62 \rangle (p_1 + p_2 + p_6)^2$$

$$\rightarrow \frac{-2 [35]^2 \langle 62 \rangle^4}{\langle 61 \rangle \langle 12 \rangle [34] [45] \langle 62 \rangle} \cdot \frac{1}{(p_1 + p_2 + p_6)^2 [51 p_6 + p_1 12] \langle 61 p_1 + p_2 3 \rangle}$$

Nima Problem Set 2

① Cross ratios: n points $x_1, \dots, x_n \in \mathbb{P}^1$

$$[x_1: y_1] \quad [x_2: y_2]$$

$$[t_1: x_1: t_2: y_1]$$

and action of $SL(2)$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

$$\begin{pmatrix} 1 \\ y/x \end{pmatrix} \begin{pmatrix} 1 \\ \frac{cx+dy}{ax+by} \end{pmatrix} \xrightarrow{\text{same?}} \frac{y}{x} = \frac{cx+dy}{ax+by} \rightarrow axy+by^2 = cx^2+dyxy$$

$$(a-d)xy+by^2-cx^2=0$$

$$ad-bc=1 \Rightarrow b = \frac{ad-1}{c}$$

$$(SL(2)) \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix} \begin{pmatrix} t_1 & t_2 & \dots & t_n \end{pmatrix}$$

So this space is $2n - n - 3 = n - 3$ dimensional

The $SL(2)$ invariant tells us all invariants we consider must be functions of the bracket

$$(ij) = \epsilon_{ab} x_i^a x_j^b$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \rightarrow \begin{pmatrix} ax_1+by_1 & ax_2+by_2 \\ cx_1+dy_1 & cx_2+dy_2 \end{pmatrix} \xrightarrow{\det} (ax_1+by_1)(cx_2+dy_2) - (ax_2+by_2)(cx_1+dy_1)$$

$$= acx_1x_2 + by_1x_2 + adx_1y_2 + bdy_1y_2 - (acx_2x_1 + bcy_2x_1 + adx_2y_1 + bdy_2y_1)$$

$$= (bc-ad)x_2y_1 + (ad-bc)x_1y_2$$

$$= x_1y_2 - x_2y_1 = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

so (ij) is invariant under action of $SL(2)$

$\rightarrow$ are all invariants of $\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix}$ (that can be built from this) under $SL(2)$ action of the form (ij) ?

We can only work with combinations of these brackets for which the Poisson bracket under rescaling each variable is zero?

For $n=3$ there are no degrees of freedom, and this is impossible with the fact that there are no Poincaré invariants we can build

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \rightarrow \begin{pmatrix} t_1 x_1 & t_2 x_2 \\ t_1 y_1 & t_2 y_2 \end{pmatrix} = t_1 t_2 (x_1 y_2 - x_2 y_1) = t_1 t_2 (x_1 y_2 - x_2 y_1)$$

$$t_1 t_2 (12) \quad t_1 t_2 (13) \quad t_1 t_2 (23) = \frac{(12)(23)(13)}{?}$$

$$2 \cdot 3 - 3 - 3 = 0 - \text{dim}$$

For $n=4$: $2 \cdot 4 - 4 - 3 = 8 - 4 - 3 = 1 - \text{dim}$

$$[12; 34] = \frac{(12)(34)}{(23)(41)} \rightarrow \text{2, 4, 3, 1 possible but by symmetry } [a,b;c,d] \text{ does the order matter? yes}$$

$$[12; 43] = \frac{(12)(43)}{(24)(31)} \rightarrow = [21; 34]$$

$$[21; 43] = \frac{(21)(43)}{(14)(32)}$$

$$[34; 12] = \frac{(34)(12)}{(41)(23)}$$

same

there are the reciprocals from $[1234]$

$$\begin{aligned} [1234] &= [2143] = [3412] = [4321] = \frac{(12)(34)}{(23)(41)} \\ [1243] &= [2134] = [4312] = [3421] = \frac{(12)(43)}{(24)(31)} \\ [1324] &= [3142] = [2413] = [4231] = \frac{(13)(24)}{(32)(41)} \\ [1342] &= [3124] = [4213] = [2431] = \frac{(13)(42)}{(34)(21)} \\ [1423] &= [4132] = [2314] = [3241] = \frac{(14)(23)}{(42)(31)} \\ [1432] &= [4123] = [3214] = [2341] = \frac{(14)(32)}{(43)(21)} \end{aligned}$$

reciprocals

How many choices of determinants?

$$\binom{4}{2} = \frac{4 \cdot 3}{2} = 6$$

$$(12)(13)(14)(23)(24)(34)$$

How many colors of $[234]$ do we have? $(12)(34), (13)(24), (14)(23) : 3$

and since we can't have the same by order (often it equals one), we have $3 \cdot 2 = 6$ possibilities

Prob: $(13)(24) = (12)(34) + (23)(41)$

$$\text{So } [1234] = \frac{(13)(24) - (23)(41)}{(12)(34)} = \frac{(13)(24)}{(12)(34)} + 1 \Rightarrow -[324] + 1 \Rightarrow [324] = [1234] - 1$$

$$= \frac{(12)(34)}{(12)(34) - (13)(24)} \rightarrow 1 - \frac{(13)(24)}{(12)(34)} = \frac{1}{[1234]} \Rightarrow [1342] = 1 - \frac{1}{[1234]} = \frac{[1234] - 1}{[1234]}$$

and of course $[1432] = \frac{1}{[1234]}$

redo:

$$[1234] = \frac{(12)(34)}{(23)(41)} = \frac{(12)(34)}{(12)(34) - (13)(24)}$$

$$[342] = \frac{(13)(42)}{(34)(21)} = \frac{(13)(-24)}{(34)(-12)} = \frac{(13)(24)}{(12)(34)}$$

$$[1243] = \frac{[1234]}{[1234] - 1}$$

Conveniently give for the two ad. $SL(2)$ action

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ by } SL(2)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} ax_1 + by_1 = 1 \\ cx_1 + dy_1 = 0 \end{cases} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} ax_4 + by_4 = 0 \\ cx_4 + dy_4 = 1 \end{cases}$$

$$\begin{aligned} ax_1 + by_1 &= 1 \rightarrow (-\frac{by_4}{x_4})x_1 + by_1 = 1 \rightarrow b(y_1 - \frac{y_4 x_1}{x_4}) = 1 \\ ax_4 + by_4 &= 0 \rightarrow a = -\frac{by_4}{x_4} \end{aligned}$$

$$b = \frac{1}{y_1 x_4 - y_4 x_1} = \frac{x_4}{x_4 y_1 - x_1 y_4}$$

$$= -\left(\frac{x_4}{x_4 y_1 - x_1 y_4}\right) \cdot \frac{y_4}{x_4}$$

$$= -\frac{y_4}{x_4 y_1 - x_1 y_4}$$

(cf p20)

By symm: $c = \frac{x_1}{x_1 y_4 - x_4 y_1}, d = -\frac{y_1}{x_1 y_4 - x_4 y_1}$

Alternatively, we only need the "0" equations: $cx_1 + dy_1 = 0 \rightarrow c = -d \frac{y_1}{x_1}$
 $ax_4 + by_4 = 0 \rightarrow a = -b \frac{y_4}{x_4}$

$$\begin{pmatrix} -b \frac{y_4}{x_4} & b \\ -d \frac{y_1}{x_1} & d \end{pmatrix} \xrightarrow{\det=1} -bd \frac{y_4}{x_4} - b(-d \frac{y_1}{x_1})$$

$$= bd \left(\frac{y_1}{x_1} - \frac{y_4}{x_4} \right) = 1$$

$$= bd \left(\frac{y_1 x_4 - x_1 y_4}{x_1 x_4} \right) = 1 \Rightarrow b = \frac{1}{d} \frac{x_1 x_4}{y_1 x_4 - x_1 y_4}$$

$$\begin{pmatrix} -\frac{1}{d} \frac{x_1 y_4}{x_1} & \frac{1}{d} \frac{x_4 y_4}{x_1} \\ -d \frac{y_1}{x_1} & d \end{pmatrix} \rightarrow \text{Can we use } d \text{ to } \frac{1}{d} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \text{ so that } x_3 = y_3?$$

$$\rightarrow \begin{pmatrix} -\frac{1}{d} \frac{x_1 x_3 y_4}{x_1} + \frac{1}{d} \frac{x_1 x_4 y_3}{x_1} \\ -d \frac{x_3 y_1}{x_1} + d y_3 \end{pmatrix} \text{ with: } -\frac{1}{d} \frac{x_1 x_3 y_4}{x_1} + \frac{1}{d} \frac{x_1 x_4 y_3}{x_1} = -d \frac{x_3 y_1}{x_1} + d y_3$$

$$\frac{x_1}{x_1} (-x_3 y_4 + x_4 y_3) = d^2 (y_3 - \frac{x_3 y_1}{x_1})$$

$$\frac{x_1}{x_1} (43) = d^2 \left(\frac{x_1 y_3 - x_3 y_1}{x_1} \right)$$

$$\frac{(43)}{(41)} x_1^2 = d^2 (13)$$

$$d^2 = \frac{(43)}{(41)(13)} x_1^2$$

$$d = \pm \sqrt{\frac{(43)}{(41)(13)}} x_1$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & z & 1 \end{pmatrix}$$

Compute all the different brackets in this group and verify their relationship to each other

$$[12, 34] = \frac{(12)(34)}{(23)(41)} = \frac{z \cdot 1}{(1-z)(-1)} = \frac{-z}{1-z} = \frac{z}{z-1}$$

$$[14, 32] = \frac{(14)(32)}{(43)(21)} = \frac{z-1}{z} \quad \text{reciprocal (check: } \frac{1 \cdot (z-1)}{(-1)(-z)})$$

$$[12, 43] = \frac{1}{[13, 42]} = \frac{(12)(43)}{(24)(31)} = \frac{z(-1)}{1(-1)} = z = \frac{1}{[13, 42]}$$

$$[13, 24] = \frac{1}{[14, 23]} = \frac{(13)(24)}{(32)(41)} = \frac{(1)(-1)}{(z-1)(1)} = \frac{1}{z-1} = \frac{1}{[14, 23]}$$

check: $[1324] = [1234] - 1$, $[1342] = \frac{[1234] - 1}{[1234]}$

$$[1234] - 1 = \frac{z}{z-1} - 1 = \frac{z - (z-1)}{z-1} = \frac{1}{z-1} = [1342]$$

$$\frac{[1234] - 1}{[1234]} = \frac{\frac{z}{z-1} - 1}{\frac{z}{z-1}} = \frac{\frac{1}{z-1}}{\frac{z}{z-1}} = \frac{1}{z} = [1342]$$

2) The Poincaré Sphere

Points on 2-sphere can also be associated with null 4-vector $p^\mu = (1, \vec{n})$, $p^2 = 0$ where $\vec{n}$ is a unit vector.
 we can write $p_\alpha = \lambda \tilde{\lambda}_\alpha$ where we identify $\tilde{\lambda} = \lambda^\dagger$ so that we have a real 4-momentum in Minkowski space. (and thus a real 2-sphere)
 This identification manifests the action of $SL(2, \mathbb{C})$ on the 2-sphere. * If acts on λ and $\tilde{\lambda}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \tilde{\lambda} \end{pmatrix} \quad \text{stereographic coordinates?}$$

$$\text{Invariant ratios: } c = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 41 \rangle}, \quad \tilde{c} = \frac{[12][34]}{[23][41]}$$

Find expressions for $c + \tilde{c}$ and $c\tilde{c}$ in terms of the dot products $p_i \cdot p_j = (1 - \hat{n}_i \cdot \hat{n}_j)$

$$c + \tilde{c} = \frac{\langle 12 \rangle \langle 34 \rangle [23][41] + [12][34] \langle 23 \rangle \langle 41 \rangle}{\langle 23 \rangle \langle 41 \rangle [23][41]} = \frac{4}{(p_4 \cdot p_1)(p_2 \cdot p_3)} \quad ?$$

$$p_i \cdot p_j = 1 - \hat{n}_i \cdot \hat{n}_j = [ij] \langle ij \rangle$$

$$(p_i + p_j)^2 = p_i^2 + 2p_i \cdot p_j + p_j^2 = 2p_i \cdot p_j$$

$$(\lambda_i^\dagger \tilde{\lambda}_i + \lambda_j^\dagger \tilde{\lambda}_j)^2$$

$$N = [41] \langle 14 \rangle$$

probably need to use ϕ trick as per a)

or use $\tilde{\lambda} = \lambda^\dagger$

(p20 BH: for real momenta $[pq]^\dagger = \langle q p \rangle$)

$$\text{so } \tilde{c} = \left(\frac{\langle 21 \rangle \langle 43 \rangle}{\langle 23 \rangle \langle 41 \rangle} \right)^*$$

$$c\tilde{c} = \frac{\langle 12 \rangle \langle 34 \rangle [12][34]}{\langle 23 \rangle \langle 41 \rangle [23][41]} = \frac{(p_1 p_2)(p_3 p_4)}{(p_2 p_3)(p_1 p_4)}$$

$$(z - c)(z - \tilde{c}) = z^2 - (c + \tilde{c})z + c\tilde{c}$$

$$= z^2 - (c + \tilde{c})z + c\tilde{c} = 0$$

$2\text{Re } c$?

$$[12][34] = [13][24] + [14][23]$$

$$N = \langle 12 \rangle \langle 34 \rangle [23][41] + \langle 23 \rangle \langle 41 \rangle [13][24] + [14][23]$$

$$= \langle 12 \rangle \langle 34 \rangle [23][41] + \langle 23 \rangle \langle 41 \rangle [13][24] + \langle 23 \rangle \langle 41 \rangle [32][14]$$

$$\langle 12 \rangle \langle 34 \rangle [23][41] + (\langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 42 \rangle) [13][24] + [32][14]$$

$$= \langle 12 \rangle \langle 34 \rangle [23][41] + \langle 12 \rangle \langle 34 \rangle [13][24] + \langle 13 \rangle \langle 42 \rangle [13][24] + \langle 13 \rangle \langle 42 \rangle [32][14] + \langle 13 \rangle \langle 42 \rangle [32][14]$$

$$\langle 12 \rangle \langle 34 \rangle [12][34] - [13][24] + [12][34] (\langle 12 \rangle \langle 34 \rangle - \langle 13 \rangle \langle 42 \rangle)$$

$$= \langle 12 \rangle \langle 34 \rangle [12][34] + [12][34] \langle 12 \rangle \langle 34 \rangle - \langle 12 \rangle \langle 34 \rangle [13][24] - [12][34] \langle 13 \rangle \langle 42 \rangle$$

$$\langle 12 \rangle \langle 34 \rangle [23][41] + (\langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 42 \rangle)$$

$$c + c^* = 2 \operatorname{Re}(c)$$

$$\text{What's } \operatorname{Re} \langle 12 \rangle = 1, 1_2$$

$$\text{by } \langle 23 \rangle \langle 41 \rangle = \frac{z}{z-1}$$

$$\frac{z}{z-1} + \left(\frac{z}{z-1}\right)^* = \frac{z}{z-1} + \frac{\bar{z}}{\bar{z}-1}$$

$$\frac{a+ib}{a+ib-1} = \frac{a}{(a-1)+ib} + \frac{ib}{(a-1)+ib}$$

$$= \frac{a(a-1) - iab + ib(a-1) - b^2}{(a-1)^2 - b^2} = \frac{a(a-1) - b^2 + i(b(a-1) - ab)}{a^2 - 2a + 1 - b^2} = \frac{a(a-1) - b^2 - i}{(a-1)^2 - b^2} \rightarrow \text{real part is } \frac{a(a-1) - b^2}{(a-1)^2 - b^2}$$

Since real, all numbers are real, so $t=1$

$$\langle 12 \rangle = \frac{1}{(1-z_1)(1-z_2)} \det \begin{pmatrix} 1-z_1 & (1-z_2) \\ -x_1-iy_1 & -x_2-iy_2 \end{pmatrix} = \frac{1}{(1-z_1)(1-z_2)} \left[(1-z_1)(-x_2-iy_2) - (1-z_2)(-x_1-iy_1) \right]$$

$$= \frac{1}{(1-z_1)(1-z_2)} \left[-x_2 - iy_2 + x_2 z_1 + iy_2 z_1 - (-x_1 - iy_1 + x_1 z_2 + iy_1 z_2) \right]$$

$$= \frac{1}{(1-z_1)(1-z_2)} \left[-x_2 - iy_2 + x_2 z_1 + iy_2 z_1 + x_1 + iy_1 - x_1 z_2 - iy_1 z_2 \right]$$

$$= \frac{(x_1 - x_2 - x_1 z_2 + x_2 z_1 + i(y_1 - y_2 + y_2 z_1 - y_1 z_2))}{(1-z_1)(1-z_2)}$$

$$c + \tilde{c} = \langle 12 \rangle \langle 34 \rangle [23] [41] + \langle 23 \rangle \langle 41 \rangle [12] [34]$$

$$\langle 23 \rangle \langle 41 \rangle [23] [41]$$

$$N = (\langle 13 \rangle \langle 24 \rangle - \langle 23 \rangle \langle 14 \rangle) [23] [41] + (\langle 12 \rangle \langle 34 \rangle - \langle 13 \rangle \langle 24 \rangle) [12] [34]$$

$$= \langle 13 \rangle \langle 24 \rangle [23] [41] - \langle 23 \rangle \langle 14 \rangle [23] [41] + \langle 12 \rangle \langle 34 \rangle [12] [34] - \langle 13 \rangle \langle 24 \rangle [12] [34]$$

$$+ (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4)$$

$$\langle 13 \rangle \langle 24 \rangle ([23] [41] - [12] [34]) \leftarrow$$

$$[13] [42]$$

$$= -(p_1 \cdot p_3)(p_2 \cdot p_4)$$

$$\text{SO } c + \tilde{c} = \frac{p_1 \cdot p_2 \cdot p_3 \cdot p_4 + p_2 \cdot p_3 \cdot p_1 \cdot p_4 - p_1 \cdot p_3 \cdot p_2 \cdot p_4}{p_2 \cdot p_3 \cdot p_1 \cdot p_4} = \frac{p_1 \cdot p_2 \cdot p_3 \cdot p_4}{p_1 \cdot p_4 \cdot p_2 \cdot p_3} - \frac{p_1 \cdot p_3 \cdot p_2 \cdot p_4}{p_2 \cdot p_3 \cdot p_1 \cdot p_4} + 1$$

$$\text{Quadratic Equations: } z^2 - \left(\frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 4 \cdot 2 \cdot 3} - \frac{1 \cdot 3 \cdot 2 \cdot 4}{2 \cdot 3 \cdot 1 \cdot 4} + 1 \right) z + \frac{1 \cdot 2 \cdot 3 \cdot 4}{2 \cdot 3 \cdot 1 \cdot 4}$$

(cont. p228)

Problem (2) cont. from p221

Show that $C - \bar{C}$ is proportional to the parity-odd defect $\varepsilon_{\mu_1 \dots \mu_4} p_1^{\mu_1} \dots p_4^{\mu_4}$

$$C - \bar{C} = \frac{\langle 12 \rangle \langle 34 \rangle [23] [41] - \langle 23 \rangle \langle 41 \rangle [12] [34]}{\langle 23 \rangle \langle 41 \rangle [23] [41]}$$

$$N = \langle 13 \rangle \langle 24 \rangle - \langle 23 \rangle \langle 14 \rangle [23] [41] - \langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle [12] [34]$$

$$= \langle 13 \rangle \langle 24 \rangle [23] [41] - \langle 23 \rangle \langle 14 \rangle [23] [41] - \langle 12 \rangle \langle 34 \rangle [12] [34] + \langle 13 \rangle \langle 24 \rangle [12] [34]$$

$$\downarrow \quad \quad \quad +2 \cdot 3 \cdot 1 \cdot 4 \quad \quad \quad -1 \cdot 2 \cdot 3 \cdot 4$$

$$\langle 13 \rangle \langle 24 \rangle ([23] [41] + [12] [34]) \leftarrow \text{this doesn't work}$$

Soln: $\langle 13 \rangle \langle 24 \rangle = \langle 12 \rangle \langle 34 \rangle + \langle 23 \rangle \langle 14 \rangle$
 $[13] [24] = [12] [34] - [23] [41]$

$$\varepsilon_{\mu_1 \dots \mu_4} p_1^{\mu_1} \dots p_4^{\mu_4} = \det \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hat{n}_1 & \hat{n}_2 & \hat{n}_3 & \hat{n}_4 \end{pmatrix} = (234) - (134) + (124) - (123)$$

What's a determinant in spinor helicity?

How do we go from $\hat{n}_i$ to $\lambda_i, \tilde{\lambda}_i$?

Is $C - \bar{C}$ completely antisymmetric in 1, 2, 3, 4? Denominator is $p_2 \cdot p_3 p_1 \cdot p_4$ symmetric.

$$1 \leftrightarrow 2: \frac{\langle 12 \rangle \langle 34 \rangle [23] [41] - \langle 23 \rangle \langle 41 \rangle [12] [34]}{\langle 23 \rangle \langle 41 \rangle [23] [41]}$$

$$\frac{\langle 21 \rangle \langle 34 \rangle [13] [42] - \langle 43 \rangle \langle 12 \rangle [21] [34]}{\langle 13 \rangle \langle 42 \rangle [13] [42]} = \frac{\langle 21 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 42 \rangle} - \frac{[21] [34]}{[13] [42]}$$

Can we go further like in problem 1? So $\langle 12, 34 \rangle = \frac{z}{z-1}$, $[12, 34] = \left(\frac{z}{z-1}\right)^*$

$$\text{nd } C - \bar{C} = \frac{z}{z-1} - \frac{\bar{z}}{\bar{z}-1} = \frac{z(\bar{z}-1) - \bar{z}(z-1)}{(z-1)(\bar{z}-1)} = \frac{z\bar{z}-1 - z\bar{z}+1}{(z-1)(\bar{z}-1)} = \frac{0}{(z-1)(\bar{z}-1)}$$

$$\text{Is } (z-1)(\bar{z}-1) = z\bar{z} - \bar{z} + z - 1$$

$$\text{np: } (z-1)(\bar{z}-1) = (a+ib-1)(a-ib-1) = ((a-1)+ib)((a-1)-ib) = (a-1)^2 + b^2$$

$$(\bar{z}-1)(z-1) = (a-ib-1)(a+ib-1) = 11$$

$$= \frac{z\bar{z} - z - \bar{z} + 1}{|z-1|^2} = \frac{1-z}{|z-1|^2} = \frac{\frac{x}{|x|^2}}{1-\frac{x}{|x|^2}} = \frac{1}{1-x}$$

$$\text{is that negative? } \frac{1-a-ib}{(a-1)^2 + b^2}$$

$$\text{Now do } \langle 234 \rangle - [21, 34] = z - \bar{z} \leftarrow \begin{matrix} a+ib - (a-ib) \\ = 2ib \end{matrix}$$

$\frac{z}{z-1}$, $\bar{z}$, $\frac{1}{z-1}$ and their reciprocals

$$z - \bar{z}, \text{ vs } \frac{z}{z-1} - \left(\frac{\bar{z}}{\bar{z}-1}\right)^* \text{ vs } \frac{1}{z-1} - \left(\frac{1}{\bar{z}-1}\right)^*$$

$$\frac{1}{(a-1)+ib} - \frac{1}{(a-1)-ib}$$

$$\frac{(a-1)-ib - ((a-1)+ib)}{(a-1)^2 + b^2} = \frac{(a-1) - (a-1) - 2ib}{(a-1)^2 + b^2} = \frac{-2ib}{(a-1)^2 + b^2} = \frac{-2i \operatorname{Im}(z)}{|z-1|^2}$$

Remember: up to a constant (proportional) is $|z-1|^2$ a constant?

Is it even to cancel $\langle \rangle, []$ to p^u or p^u to $\langle \rangle []$?
 probably this one

Anyway, if I can show that $c - \bar{c}$ is completely antisymmetric, since it only depends on $\langle i \text{ and } i \rangle$ which are in p_i^u , then ready to call it a day on this problem.

Back to $1 \leftrightarrow 2$: $(c - \bar{c})_{1234} \rightarrow (c - \bar{c})_{2134}$

I am going to use the velocity on p217:

$$[2134] = [1243] = \frac{[1234]}{[1234]-1}$$

Let $[1234] = z$

$$z - \bar{z}$$

$$\frac{z}{z-1} - \left(\frac{\bar{z}}{\bar{z}-1}\right)^* = \frac{z}{z-1} - \frac{\bar{z}}{\bar{z}-1}$$

$$z^2 = c\bar{c} = \frac{p_1 p_2 p_3 p_4}{p_2 p_3 p_4 p_1}$$

$$|z-1|^2 = (c-1)(\bar{c}-1) = c\bar{c} - (c+\bar{c}) + 1 = \frac{1234}{2314} -$$

$$\boxed{z = c}$$

$$\text{so is } |c - \bar{c}|^2 \cdot (c - \bar{c}) = \det(p_1 p_2 p_3 p_4)$$

constant of proportionality?

$$1 \leftrightarrow 2 = 3 \leftrightarrow 4$$

$$2 \leftrightarrow 3 = 4 \leftrightarrow 1$$

$$2 \leftrightarrow 4 = 1 \leftrightarrow 3$$

$$\text{Let's try } 2 \leftrightarrow 3: \frac{(13)(24)}{(32)(41)} = z - 1 \text{ so } (z-1) - (\bar{z}-1) = z - \bar{z}$$

Did I make a mistake? check:

$$\frac{(12)(34)}{(23)(41)} \rightarrow \frac{(13)(24)}{(32)(41)} \text{ plus: } (13)(24) = (12)(34) + (23)(14)$$

$$= \frac{(12)(34) + (23)(14)}{(32)(41)} = \frac{(12)(34)}{(32)(41)} + \frac{(23)(14)}{(32)(41)} = -z + 1$$

so actually: $1 \leftrightarrow 2: [2134] = [1293] = \frac{1}{[1342]} = \frac{1}{1 - \frac{1}{z}} = \frac{z-1}{z} = \frac{z}{z-1}$

$$\text{so } (c - \bar{c})_{2134} = \frac{z}{z-1} - \left(\frac{\bar{z}}{\bar{z}-1}\right)^* \text{ check}$$

$$2 \leftrightarrow 4, 1 \leftrightarrow 3 \text{ is } \frac{1}{[1234]} \text{ so } \frac{1}{z} - \frac{1}{\bar{z}} = \frac{\bar{z} - z}{z\bar{z}}: \boxed{\frac{z - \bar{z}}{|z|^2} \text{ } 2 \leftrightarrow 4, 1 \leftrightarrow 3}$$

Remark: $\sum_{\mu_1, \dots, \mu_4} p_1^{\mu_1} \dots p_4^{\mu_4}$: change $p_i^{\mu_i}$ to $\lambda_i \tilde{\lambda}_i = (p_i)_{\alpha\dot{\alpha}}$

$$(234) + (134) + (124) - (123)$$

$$\det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

Can we find the value?

$$\frac{1}{\sqrt{1-z}} \begin{pmatrix} 1-z \\ -x-y \end{pmatrix} \frac{1}{\sqrt{1-z}} \begin{pmatrix} 1-z \\ -x+y \end{pmatrix} \quad t=1 \text{ since real?}$$

should get

$$\begin{aligned} & x_2(y_3 z_4 - z_3 y_4) + x_3(y_4 z_2 - y_2 z_4) + x_4(y_2 z_3 - z_2 y_3) \\ & - x_1(y_3 z_4 - z_3 y_4) + x_3(y_4 z_1 - y_1 z_4) + x_4(y_1 z_3 - z_1 y_3) \\ & + x_1(y_2 z_4 - z_2 y_4) + x_2(y_4 z_1 - y_1 z_4) + x_4(y_1 z_2 - z_1 y_2) \\ & - x_1(y_2 z_3 - z_2 y_3) + x_2(y_3 z_1 - y_1 z_3) + x_3(y_1 z_2 - z_1 y_2) \end{aligned}$$

$$\text{Now take } \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 41 \rangle}$$

$$\begin{aligned} \langle 12 \rangle &= \lambda_1 \lambda_2 = \frac{1}{\sqrt{(1-z)(1-\bar{z})}} \left[(1-\bar{z})(-x_2+y_2) - (1-z)(-x_1+y_1) \right] \\ &= \frac{1}{\sqrt{(1-z)(1-\bar{z})}} \left[(1-\bar{z})(x_1+y_1) - (1-z)(x_2+y_2) \right] \\ &= \frac{1}{\sqrt{(1-z)(1-\bar{z})}} \left(x_1+y_1 - x_1\bar{z}_2 - y_1\bar{z}_2 - [x_2+y_2 - x_2\bar{z}_1 - y_2\bar{z}_1] \right) \\ &= \frac{1}{\sqrt{(1-z)(1-\bar{z})}} \left(x_1-x_2 + \underbrace{x_2\bar{z}_1 - x_1\bar{z}_2}_{(\bar{z}x)_{12}} + i(y_1-y_2 + \underbrace{y_2\bar{z}_1 - y_1\bar{z}_2}_{(\bar{z}y)_{12}}) \right) \end{aligned}$$

-1 for [12]

$$\langle 34 \rangle = \frac{1}{\sqrt{(1-\bar{z})(1-z)}} (x_3-x_4 + x_4\bar{z}_3 - x_3\bar{z}_4) + i(y_3-y_4 + y_4\bar{z}_3 - y_3\bar{z}_4)$$

$$\langle 23 \rangle = \frac{1}{\sqrt{(1-\bar{z})(1-z)}} (x_2-x_3 + x_3\bar{z}_2 - x_2\bar{z}_3) + i(y_2-y_3 + y_3\bar{z}_2 - y_2\bar{z}_3)$$

$$\langle 41 \rangle = \frac{1}{\sqrt{(1-\bar{z})(1-z)}} (x_4-x_1 + x_1\bar{z}_4 - x_4\bar{z}_1) + i(y_4-y_1 + y_1\bar{z}_4 - y_4\bar{z}_1)$$

Plug in and out

$$\frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 41 \rangle} = \frac{(x_1-x_2 + (\bar{z}x)_{12} + i(y_1-y_2 + (\bar{z}y)_{12})) (x_3-x_4 + (\bar{z}x)_{34} + i(y_3-y_4 + (\bar{z}y)_{34}))}{(x_2-x_3 + (\bar{z}x)_{23} + i(y_2-y_3 + (\bar{z}y)_{23})) (x_4-x_1 + (\bar{z}x)_{41} + i(y_4-y_1 + (\bar{z}y)_{41}))}$$

$$- \frac{[12][34]}{[23][41]} = - \frac{(x_1-x_2 + (\bar{z}x)_{12} - i(y_1-y_2 + (\bar{z}y)_{12})) (x_3-x_4 + (\bar{z}x)_{34} - i(y_3-y_4 + (\bar{z}y)_{34}))}{(x_2-x_3 + (\bar{z}x)_{23} - i(y_2-y_3 + (\bar{z}y)_{23})) (x_4-x_1 + (\bar{z}x)_{41} - i(y_4-y_1 + (\bar{z}y)_{41}))} \rightarrow \text{plug into Mathematica?}$$

Mathematical
Simplification

$$\frac{(a_1 + ia_2)(b_1 + ib_2)}{(c_1 + ic_2)(d_1 + id_2)} - \frac{(a_1 + ia_2)(b_1 - ib_2)}{(c_1 + ic_2)(d_1 - id_2)} = \frac{(a_1 + ia_2)(b_1 + ib_2)(c_1 - ic_2)(d_1 - id_2) - (a_1 + ia_2)(b_1 - ib_2)(c_1 + ic_2)(d_1 + id_2)}{(c_1^2 + c_2^2)(d_1^2 + d_2^2)}$$

$$0 = (1 + i)(1 - i)(1 - i) = 0 = (1 - i)(1 + i) = 0$$

$$\frac{(1 - i)}{(1 + i)} = \frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

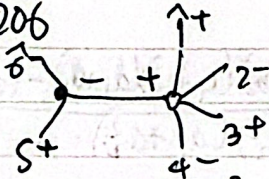
$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

$$\frac{(1 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{(1 - i)^2}{1 - i^2} = \frac{1 - 2i + i^2}{1 - (-1)} = \frac{1 - 2i - 1}{2} = \frac{-2i}{2} = -i$$

BCPW out from p 206

Next diagram: (B)



$$\frac{\langle 6 | \hat{I}^3 | 3 \rangle}{\langle 65 | \langle 15 \rangle} \cdot \frac{1}{[56] \langle 56 \rangle} \cdot \frac{\langle 24 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \langle 11 \rangle$$

$$p_{36}^2 = [56] \langle 56 \rangle = 0 \rightarrow \langle 56 \rangle [5(\delta + \epsilon)] = 0$$

$$\langle 56 \rangle ([5\delta] + \epsilon[51]) = 0$$

$$\epsilon = -\frac{[5\delta]}{[51]}$$

$$\langle 1' \rangle = \langle 1 - \epsilon \langle 6 \rangle = \langle 1 + \frac{[5\delta]}{[51]} \langle 6 \rangle = \frac{[51] \langle 1 \rangle + [5\delta] \langle 6 \rangle}{[51]} = \frac{[51] (p_1 + p_6)}{[51]}$$

We know $p_1 = p_5 + p_6$

$$\lambda_1 \tilde{\lambda}_1 = \lambda_5 \tilde{\lambda}_5 + \lambda_6 \tilde{\lambda}_6 \xrightarrow{\text{solve for } \tilde{\lambda}_1} \lambda_1 \tilde{\lambda}_5 = \lambda_5 \tilde{\lambda}_5 + \lambda_6 (\tilde{\lambda}_6 + \frac{[5\delta]}{[51]} \tilde{\lambda}_1)$$

$$= \lambda_5 \tilde{\lambda}_5 + \lambda_6 \tilde{\lambda}_6 - \frac{[5\delta]}{[51]} \lambda_6 \tilde{\lambda}_1$$

$$\text{solve: } [56] \tilde{\lambda}_1 - [16] \tilde{\lambda}_5 + [15] \tilde{\lambda}_6 = 0$$

$$[51] \tilde{\lambda}_6 - [56] \tilde{\lambda}_1 = -[16] \tilde{\lambda}_5$$

$$= [61] \tilde{\lambda}_5$$

$$\frac{[51] \lambda_6 \tilde{\lambda}_6 - [56] \lambda_6 \tilde{\lambda}_1}{[51]} = \frac{\lambda_6 ([51] \tilde{\lambda}_6 - [56] \tilde{\lambda}_1)}{[51]}$$

$$= \frac{[61]}{[51]} \tilde{\lambda}_5 \lambda_6$$

$$\Rightarrow \lambda_1 = \lambda_5 + \frac{[61]}{[51]} \lambda_6$$

$$= \frac{[51] \lambda_5 + [61] \lambda_6}{[51]} = \frac{(p_5 + p_6)}{[51]}$$

$$\langle \lambda_1 6 \rangle = \langle 56 \rangle$$

$$\langle \lambda_1 5 \rangle = \frac{[61]}{[51]} \langle 65 \rangle$$

$$\text{so } \frac{\langle 6 | \hat{I}^3 | 3 \rangle}{\langle 65 \rangle \langle 15 \rangle} \cdot \frac{1}{[56] \langle 56 \rangle} = \frac{\langle 56 \rangle^3}{\langle 65 \rangle [61] \langle 65 \rangle [56] \langle 56 \rangle} \cdot \frac{1}{[51]} = \frac{[51]}{[61] [56]}$$

$$\langle 12 \rangle \langle 41 \rangle \langle 11 \rangle$$

$$= \frac{[5(p_1 + p_6 | 2)]}{[51]} \frac{\langle 4 | p_5 + p_6 | 1 \rangle}{[51]} \frac{[1(p_5 + p_6)(p_1 + p_6) | 5]}{[51]^2}$$

$$[1(p_5 + p_6)(p_1 + p_6) | 5] = [15] \langle 51 \rangle [15] + [15] \langle 56 \rangle [65] + [16] \langle 61 \rangle [05]$$

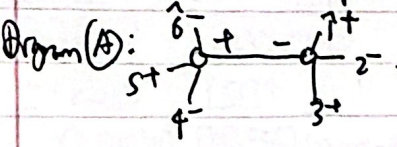
$$= [15] (-p_1 \cdot p_5 + p_5 \cdot p_6 - p_1 \cdot p_6)$$

$$= \frac{[15]}{2} (p_1 + p_5 + p_6)^2$$

$$\frac{[51(p_1 + p_6 | 2)]}{[51]^4} \frac{\langle 4 | p_5 + p_6 | 1 \rangle}{[51]} \cdot \frac{1}{2} \cdot \frac{[51]}{2} (p_1 + p_5 + p_6)^2$$

$$\textcircled{1} \frac{[35]^2 \langle 26 \rangle^4}{\langle 61 \rangle \langle 42 \rangle \langle 34 \rangle \langle 45 \rangle (p_1 + p_2 + p_3)^2 \langle 61 p_1 p_2 | 3 | 51 p_2 p_1 | 2 \rangle}$$

$$\rightarrow = \frac{[81]}{[61][56]} \cdot \frac{\langle 24 \rangle^2}{\langle 23 \rangle \langle 34 \rangle} \frac{2 \cdot [51]^4}{[51 p_1 + p_6 | 2 \rangle \langle 4 | p_1 + p_6 | 1 \rangle (p_1 + p_2 + p_6)^2} [51] = \frac{\textcircled{2} \langle 24 \rangle^2 [51]^4}{[61][56] \langle 23 \rangle \langle 34 \rangle [51 p_1 + p_6 | 2 \rangle \langle 4 | p_1 + p_6 | 1 \rangle (p_1 + p_2 + p_6)^2}$$

Diagram (A):  = $\text{box: } \frac{\langle 61 \rangle^2}{\langle 61 \rangle \langle 14 \rangle \langle 45 \rangle \langle 56 \rangle}$ or $\frac{[51]^2}{[65][14][45][56]}$

Int: $\frac{\langle 12 \rangle^2}{\langle 11 \rangle \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}$ or $\frac{[13]^2}{[11][12][23][31]}$

$$\frac{1}{p_1^2} = \frac{1}{(p_1^2 + p_2^2 + p_3^2)} = \frac{1}{(p_1^2 + p_2^2 + p_3^2)^2} = \frac{2}{p_1 p_2 + p_1 p_3 + p_2 p_3}$$

Example: don't have to deal with shift

$$\frac{\langle 61 \rangle^2}{\langle 61 \rangle \langle 14 \rangle \langle 45 \rangle \langle 56 \rangle} \cdot \frac{[13]^2}{[11][12][23][31]} \frac{1}{(p_1^2 + p_2^2 + p_3^2)^2}$$

Shift: $\hat{1}, \hat{1}$

What's the λ comp of $\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = \hat{p}_I = \lambda_I \hat{\lambda}_I = p_1 + p_2 + p_3$

Integrals: $\langle 6\hat{1} \rangle \langle \hat{1}4 \rangle = \langle 61 \rangle \langle p_1 p_2 | 1 \rangle = \langle 61 p_1 p_2 | 1 \rangle$
 $\langle 3\hat{1} \rangle \langle \hat{1}4 \rangle = \langle 31 \rangle \langle p_1 p_2 | 4 \rangle = \langle 31 p_1 p_2 | 4 \rangle$
 $\langle 6\hat{1} \rangle \langle \hat{1}3 \rangle = \langle 61 \rangle \langle p_1 p_2 | 3 \rangle = \langle 61 p_1 p_2 | 3 \rangle$
 $\langle 4\hat{1} \rangle \langle \hat{1}3 \rangle = \langle 41 \rangle \langle p_1 p_2 | 3 \rangle = \langle 41 p_1 p_2 | 3 \rangle$

$$(1 + z\langle 6 \rangle) \langle 1 \rangle + z\langle 2 \rangle \langle 3 \rangle \langle 3 \rangle = 4\langle 4 \rangle \langle 5 \rangle \langle 5 \rangle + 6\langle 6 \rangle \langle 6 \rangle \langle 6 \rangle$$

$$\langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \langle 3 \rangle + z\langle 6 \rangle \langle 1 \rangle \langle 1 \rangle = 4\langle 4 \rangle \langle 5 \rangle \langle 5 \rangle + 6\langle 6 \rangle \langle 6 \rangle \langle 6 \rangle$$

$$\langle 6 \rangle \langle 1 \rangle \langle 1 \rangle = \langle 61 \rangle \langle 11 \rangle = \langle 61 \rangle \langle 11 \rangle + \langle 65 \rangle \langle 51 \rangle$$

$$\langle 12 \rangle \langle 26 \rangle + \langle 13 \rangle \langle 36 \rangle + z\langle 16 \rangle \langle 66 \rangle = \langle 14 \rangle \langle 46 \rangle + \langle 15 \rangle \langle 56 \rangle - z\langle 16 \rangle \langle 66 \rangle$$

$$\Rightarrow z = \frac{\langle 14 \rangle \langle 46 \rangle + \langle 15 \rangle \langle 56 \rangle - \langle 12 \rangle \langle 26 \rangle - \langle 13 \rangle \langle 36 \rangle}{2 p_1 p_6}$$

$$p_1^2 = 0: (\hat{p}_1 + p_2 + p_3)(\hat{p}_1 + p_2 + p_3) = 0$$

$$= 2(p_1 p_2 + p_1 p_3 + p_2 p_3) = 0$$

$$\langle 12 \rangle \langle 12 \rangle + \langle 13 \rangle \langle 13 \rangle + \langle 23 \rangle \langle 23 \rangle = 0, \text{ also: } [45] \langle 45 \rangle + [46] \langle 46 \rangle + [56] \langle 56 \rangle = 0$$

$$\langle 12 \rangle (\langle 1 \rangle - z\langle 6 \rangle) + \langle 13 \rangle (\langle 1 \rangle - z\langle 6 \rangle) + \langle 23 \rangle \langle 23 \rangle = 0$$

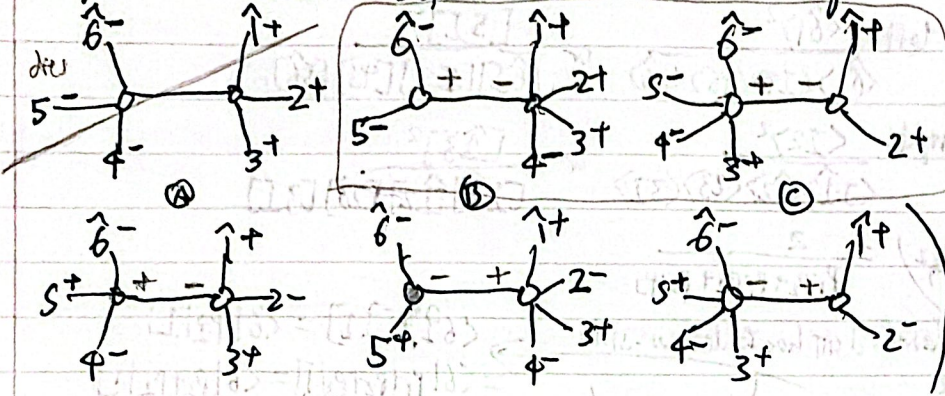
$$\langle 12 \rangle (\langle 1 \rangle - z\langle 6 \rangle) + \langle 13 \rangle (\langle 1 \rangle - z\langle 6 \rangle) + \langle 23 \rangle \langle 23 \rangle = 0$$

$$\langle 12 \rangle \langle 12 \rangle - z\langle 12 \rangle \langle 62 \rangle + \langle 13 \rangle \langle 13 \rangle - z\langle 13 \rangle \langle 63 \rangle + \langle 23 \rangle \langle 23 \rangle = 0$$

$$z = \frac{p_1 p_2 + p_1 p_3 + p_2 p_3}{2(\langle 12 \rangle \langle 62 \rangle - \langle 13 \rangle \langle 63 \rangle)}$$

$$\textcircled{1} \frac{\langle 61 \rangle^2 [13]^2}{[12][23] \langle 45 \rangle \langle 56 \rangle \langle 41 p_2 + p_3 | 1 \rangle \langle 61 p_1 + p_2 | 3 \rangle (p_1 + p_2 + p_3)^2}$$

Do A [1+2+3+4-5-6-] fully Ewg-Hung
 First, what if we did the [6] (Chell as before)? Diagram



(before)

only 2 diagrams for this one, but almost the same as before

$$\textcircled{B} \frac{\langle 65 \rangle^3}{\langle 61 \rangle \langle 51 \rangle} \cdot \frac{1}{\langle 56 \rangle \langle 66 \rangle} \cdot \frac{\langle 14 \rangle^2}{\langle 11 \rangle \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

$$\begin{aligned} p_{56}^2 &= \langle 56 \rangle \langle 66 \rangle = \langle 56 \rangle \langle 36 \rangle = 0 \\ 0 &= \langle 56 \rangle = \langle 3(6) + 2(1) \rangle = \langle 56 \rangle + 2 \langle 51 \rangle = 0 \\ z &= -\frac{\langle 56 \rangle}{\langle 51 \rangle} \end{aligned}$$

$$\begin{aligned} \langle 1 \rangle &= \langle 1 - z \langle 6 \rangle = \langle 1 + \frac{\langle 56 \rangle}{\langle 51 \rangle} \langle 6 \rangle \\ &= \frac{\langle 51 \rangle \langle 1 + \frac{\langle 56 \rangle}{\langle 51 \rangle} \langle 6 \rangle}{\langle 51 \rangle} = \frac{\langle 51 \rangle p_1 + p_6}{\langle 51 \rangle} \end{aligned}$$

$$\text{Remember } \langle 1 \rangle = \frac{(p_5 + p_6) \langle 1 \rangle}{\langle 51 \rangle} = 5 + \frac{\langle 61 \rangle}{\langle 51 \rangle} \langle 6 \rangle$$

$$\begin{aligned} \langle 61 \rangle &= \langle 65 \rangle \\ \langle 51 \rangle &= \frac{\langle 61 \rangle}{\langle 51 \rangle} \langle 56 \rangle \end{aligned}$$

$$\frac{\langle 65 \rangle^3}{\langle 61 \rangle \langle 51 \rangle} \cdot \frac{1}{\langle 56 \rangle \langle 66 \rangle} = \frac{\langle 65 \rangle^3 \langle 51 \rangle}{\langle 61 \rangle \langle 56 \rangle} \cdot \frac{1}{\langle 56 \rangle \langle 66 \rangle} \leftarrow \text{same as before}$$

$$\text{Now: } \frac{\langle 14 \rangle^2}{\langle 11 \rangle \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} = \frac{(\langle 45 \rangle + \frac{\langle 61 \rangle}{\langle 51 \rangle} \langle 46 \rangle)^2}{\frac{[1(p_5 + p_6)(p_1 + p_6) \langle 5 \rangle]}{[51][51]} \frac{[5(p_1 + p_6) \langle 2 \rangle]}{[51]} \frac{\langle 41(p_5 + p_6) \rangle}{[51]}}$$

$$\begin{aligned} 0 | p_5 p_1 + p_3 p_6 + p_6 p_1 + p_6^2 | 5 &= [5] \langle 51 \rangle [5] + [5] \langle 56 \rangle \langle 65 \rangle + [5] \langle 61 \rangle [5] \\ &= [5] (p_1 p_5 - p_3 p_6 - p_1 p_6) = -\frac{[5]}{2} (p_1 + p_3 + p_6)^2 \end{aligned}$$

$$\textcircled{C} \frac{[12]^2}{[11][21]} \cdot \frac{1}{\langle 42 \rangle \langle 12 \rangle} \cdot \frac{[13]^2}{[13][31] \langle 43 \rangle \langle 56 \rangle \langle 61 \rangle}$$

$$\begin{aligned} p_{12}^2 &= \langle 12 \rangle \langle 22 \rangle = \langle 12 \rangle \langle 12 \rangle = 0 \\ 0 &= \langle 12 \rangle = \langle (1 - z \langle 6 \rangle) \langle 2 \rangle = \langle 12 \rangle - z \langle 62 \rangle \\ z &= \frac{\langle 12 \rangle}{\langle 62 \rangle} \end{aligned}$$

$$\begin{aligned} \langle 2 \rangle &= \langle 2 + z \langle 1 \rangle = \langle 2 \rangle + \frac{\langle 12 \rangle}{\langle 62 \rangle} \langle 1 \rangle = \frac{\langle 62 \rangle \langle 2 \rangle + \langle 12 \rangle \langle 1 \rangle}{\langle 62 \rangle} \\ &= \frac{p_6 + p_1 \langle 2 \rangle}{\langle 62 \rangle} \end{aligned}$$

$$\langle 1 \rangle = \frac{(p_1 + p_2) \langle 1 \rangle}{\langle 62 \rangle} = \frac{\langle 61 \rangle}{\langle 62 \rangle} \langle 1 \rangle + \langle 2 \rangle$$

$$\begin{aligned} \langle 11 \rangle &= \langle 22 \rangle \\ \langle 21 \rangle &= \frac{\langle 61 \rangle}{\langle 62 \rangle} \langle 21 \rangle \end{aligned}$$

$$\frac{[12]^3}{[11][21]} \cdot \frac{1}{\langle 12 \rangle \langle 12 \rangle} = \frac{[12]^3 \langle 62 \rangle}{[11] \langle 61 \rangle [21] \langle 42 \rangle \langle 12 \rangle} = \frac{\langle 62 \rangle}{\langle 61 \rangle \langle 12 \rangle}$$

So far (b) we get:

$$\begin{aligned}
 & \frac{[51]}{[61][56]} \cdot \frac{(\langle 45 \rangle + \frac{[61]}{[51]} \langle 46 \rangle)}{\langle 23 \rangle \langle 34 \rangle} \cdot \frac{[51]^3}{[51] (p_1 + p_3 + p_6)^2 [5(p_1 + p_6)2] \langle 4(p_1 + p_6)1 \rangle} \\
 &= \frac{\langle 45 \rangle}{\langle 23 \rangle \langle 34 \rangle} \cdot \frac{[51]^4}{[61][56] (p_1 + p_3 + p_6)^2 [5(p_1 + p_6)2] \langle 4(p_1 + p_6)1 \rangle} + \frac{\langle 46 \rangle}{\langle 23 \rangle \langle 34 \rangle} \cdot \frac{[51]^3}{[56] (p_1 + p_3 + p_6)^2 [5(p_1 + p_6)2] \langle 4(p_1 + p_6)1 \rangle} \\
 &= \frac{[51]^3}{[56] \langle 23 \rangle \langle 34 \rangle} \cdot \frac{1}{(p_1 + p_3 + p_6)^2 [5(p_1 + p_6)2] \langle 4(p_1 + p_6)1 \rangle} \left(\frac{\langle 45 \rangle [51]}{[61]} + \langle 46 \rangle \right) \\
 &\rightarrow - \frac{[51]^3}{[61][56] \langle 23 \rangle \langle 34 \rangle} \cdot \frac{1}{(p_1 + p_3 + p_6)^2 [5(p_1 + p_6)2]} \cdot (\langle 45 \rangle + \frac{[61]}{[51]} \langle 46 \rangle)
 \end{aligned}$$

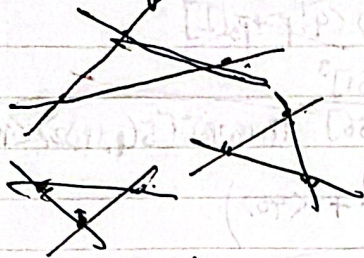
For (c): $[56][81]$

$$\begin{aligned}
 &= \frac{[5(p_1 + p_6)2]}{\langle 62 \rangle} \frac{\langle 2(p_1 + p_6)(p_1 + p_2)16 \rangle}{\langle 62 \rangle \langle 62 \rangle} \rightarrow \langle 2(p_1^2 + p_1 p_2 + p_6 p_1 + p_6 p_2)6 \rangle \\
 &= \langle 21 \rangle [12] \langle 26 \rangle + \langle 26 \rangle [61] \langle 16 \rangle + \langle 26 \rangle [62] \langle 26 \rangle \\
 &= \langle 26 \rangle (-p_1 p_2 - p_1 p_6 - p_2 p_6) \\
 &= -\langle 26 \rangle (p_1 + p_2 + p_6)^2 \\
 &= \frac{[5(p_1 + p_6)2]}{\langle 62 \rangle^2} (p_1 + p_2 + p_6)^2
 \end{aligned}$$

$$\text{so total: } - \frac{\langle 62 \rangle}{\langle 61 \rangle \langle 12 \rangle} \frac{1}{[34][45]} \frac{\langle 62 \rangle^2}{[5(p_1 + p_6)2] (p_1 + p_3 + p_6)^2} \left(\frac{[61]}{\langle 62 \rangle} [13] + [23] \right)$$

Let's see if we can get $113 + 11123 + 11193$ to have symmetric poles cancel out

Determine point on which the 3 planes (123) , (456) , (789) intersect



$$\begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} = [xyz] / 0$$

4 space: lines and 3-planes dual
pts $\updownarrow$ 2-planes dual

2 planes intersect in a line

123

What's (12)? $\epsilon_{ij} x_i^i x_j^j$ and $L_k = \epsilon_{ijk} x_i^i x_j^j$?

$$[123]: \epsilon_{ijk} x_i^j x_2^j x_3^k = \langle 123 \rangle$$

$$\left. \begin{array}{l} \langle 123 \rangle \\ \langle 456 \rangle \\ \langle 789 \rangle \end{array} \right\} = 0$$

$$\begin{array}{r} \langle 1234 \rangle \langle 789 \rangle \\ + \langle 1235 \rangle \langle 789 \rangle \\ + \langle 1236 \rangle \langle 789 \rangle \end{array}$$

completing signature (123) : $E_{IJKL} X_1^J X_2^K X_3^L = \det \begin{pmatrix} X_1^J & X_2^K & X_3^L \\ X_1^J & X_2^K & X_3^L \\ X_1^J & X_2^K & X_3^L \\ X_1^J & X_2^K & X_3^L \end{pmatrix}$

$$\begin{array}{cccccccccccccccc} \epsilon_{MAEI} & \epsilon_{ABCD} & x_1 & x_2 & x_3 & \epsilon_{EFGH} & x_4 & x_5 & x_6 & \epsilon_{IJKL} & x_7 & x_8 & x_9 \\ 11 & & & & & & & & & & & & \end{array}$$

y^m is the point: 11

$$\langle \cdot, \langle \cdot, x_1 x_2 x_3 \rangle, \langle \cdot, x_4 x_5 x_6 \rangle, \langle \cdot, x_7 x_8 x_9 \rangle \rangle =$$

Try: $X_1 \langle 23|4,5,6,7,8,9 \rangle$

 $x_2 \in \{2, 3, 4, 5, 6, 7, 8, 9\}$

73 423 45, 6, 3, 9

४५

52

76

47

42

٢٩

•

2008

Check: $\angle 789 = 0$ obviously!

$\angle 123:$

$$\langle 1237 \rangle \langle 4568 \rangle \langle 9569 \rangle$$
~~(1234)~~ ~~(1239)~~ ~~(9567)~~ ~~(9568)~~
$$+ (1237) + (238) + (9567) + (9569)$$

* 123456789101112

12383 12397 12307
12377 12313 12367

$$\frac{1}{2} \langle 1237 \rangle \langle 1239 \rangle \langle 4567 \rangle \langle 8568 \rangle$$
$$- \langle 1238 \rangle \langle 1239 \rangle \langle 9164 \rangle^2$$

✓BS!

$$\text{Wait: } \langle 4123 \rangle x_1 + \langle 5123 \rangle x_2 + \langle 6123 \rangle x_3 + \langle 7123 \rangle x_4 + \langle 8123 \rangle x_5 + \langle 9123 \rangle x_6 = 0$$

Obv:

We have solution plane for 4-D?

3 vectors in 2-D goes to 5 vectors in 4-D:

$$\langle x_2 x_3 x_4 x_5 \rangle x_1 - \langle x_1 x_3 x_4 x_5 \rangle x_2 + \langle x_1 x_2 x_4 x_5 \rangle x_3 - \langle x_1 x_2 x_3 x_5 \rangle x_4 + \langle x_1 x_2 x_3 x_4 \rangle x_5 = 0$$

(In projective space, any 5 points are linearly dependent: lie in the same 3-D space)

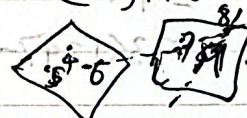
$$= x_1 \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + x_2 \begin{pmatrix} -x_3 \\ -x_4 \\ -x_5 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \langle 1456 \rangle \\ \langle 2456 \rangle \end{pmatrix}$$

$$+ \begin{pmatrix} \langle 2456 \rangle \\ \langle 3456 \rangle \end{pmatrix}$$

$$+ \begin{pmatrix} \langle 3456 \rangle \\ \langle 1456 \rangle \end{pmatrix}$$

→ ~~$\langle 4567 \rangle \langle 2345 \rangle$~~ but 45 to be the line only which $(456) \cap (789)$



$$\begin{aligned} \langle 5678 \rangle 4 - \langle 4678 \rangle 5 + \langle 4578 \rangle 6 &= \langle 4568 \rangle 7 = \langle 4567 \rangle 8 = A \\ \langle 5679 \rangle 4 - \langle 4679 \rangle 5 + \langle 4579 \rangle 6 &= \langle 4569 \rangle 7 - \langle 4567 \rangle 9 = B \end{aligned}$$

$$\Rightarrow \langle 23AB \rangle 1 - \langle 13AB \rangle 2 + \langle 12AB \rangle 3 = \langle 123B \rangle A - \langle 123A \rangle B$$

$$\begin{aligned} & \langle 4569 \rangle \langle 1237 \rangle - \langle 4567 \rangle \langle 1239 \rangle = \langle 4568 \rangle 7 - \langle 4567 \rangle 8 \\ & - (\langle 4568 \rangle \langle 1237 \rangle - \langle 4567 \rangle \langle 1238 \rangle) = \langle 4569 \rangle 7 - \langle 4567 \rangle 9 \end{aligned}$$

$$\begin{aligned} &= (\langle 4569 \rangle \langle 1237 \rangle - \langle 4567 \rangle \langle 1239 \rangle) 7 + (\langle 4567 \rangle \langle 1239 \rangle - \langle 4569 \rangle \langle 1237 \rangle) 8 \\ &+ (\langle 4567 \rangle \langle 1238 \rangle - \langle 4568 \rangle \langle 1237 \rangle) 7 + (\langle 4568 \rangle \langle 1237 \rangle - \langle 4567 \rangle \langle 1238 \rangle) 9 \end{aligned}$$

B) (12), (345), (678) in $\mathbb{P}^3$?

(12) = (345) $\cap$ (678)? $\neq$ no: use the wedge signs:

Show that this requires:

$$\langle (12) \cap (345) \cap (678) \rangle = \langle 1345 \rangle \langle 2678 \rangle - \langle 2345 \rangle \langle 1678 \rangle = 0$$

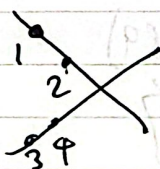
Is this sufficient?

Either L is in plane (345), or it is in plane (678), or it forms a plane with (345) $\cap$ (678) whose intersection with the other two planes is the single point $\rightarrow$ this is the case of question 1

$$\langle 2678 \rangle \langle 1345 \rangle + \langle 1678 \rangle \langle 2345 \rangle \leftarrow \text{wedge of (12) with plane 678}$$

should be $\langle 6782 \rangle$?

Exercise: Can I do



$$= 1 \langle 234 \rangle - 2 \langle 134 \rangle = 3 \langle 124 \rangle - 4 \langle 23 \rangle$$

There is a shortcut in $\mathbb{P}^2$:
that: 3D - any 4 vertices are linearly dependent.
projected in any 4 points, any can be expressed in terms of the other 3:
the 4th point lies in the plane of the other 3

X_1^I, X_2^J into a line

$\exists \epsilon_{IJK} X_1^I X_2^J$ adjoin X_1 and X_2 but it has a linear index

Can we do $X_1^I X_2^J$ more intrinsically?

It seems like we really need the ϵ :

$$\langle X_1, X_2, \dots \rangle = \epsilon_{IJK} \dots X_1^I X_2^J \leftarrow \text{this is the actual dual}$$

What's $\epsilon_{IJK} X_1^I X_2^J$?

We're in $\mathbb{P}^2$: we have $\epsilon_{IJK} = \langle \cdot, \cdot, \cdot \rangle$

$$X_1^I X_2^J = \epsilon_{IJK} X_1^I X_2^J \leftarrow \text{lines}$$

$$(12) \cap (34) =$$

We saw that (12) intersects both 345 and 678 can be written as

$$1 \langle 2345 \rangle - 2 \langle 1345 \rangle = 3 \langle 1245 \rangle \pm 4 \langle 1235 \rangle \pm 5 \langle 1234 \rangle$$

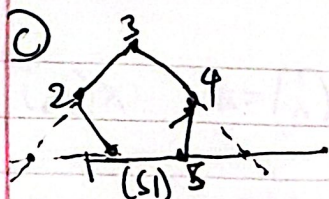
Similarly for (12) $\cap$ (678)

$$1 \langle 2678 \rangle - 2 \langle 1678 \rangle = 6 \langle 1278 \rangle \pm 7 \langle 1268 \rangle \pm 8 \langle 1267 \rangle$$

So (necessarily) condition is that this point lies on (678) for the first one, on (345) for the second one, and

$$\langle 678 \rangle \cap \langle 2345 \rangle - \langle 6782 \rangle \langle 1345 \rangle = 0 = \langle 6783 \rangle \langle 1245 \rangle + \dots$$

$$\langle 345 \rangle \cap \langle 2678 \rangle - \langle 3452 \rangle \langle 1678 \rangle = 0 = \langle 3456 \rangle \langle 1278 \rangle + \dots$$



3-buckets $\langle abc \rangle$: are we in $\mathbb{R}^3$ or $\mathbb{P}^2$?

In $\mathbb{P}^2$ can the 4 points 2, 3, 4, 5

$$[2154] = \frac{(21)(54)}{(15)(24)}$$

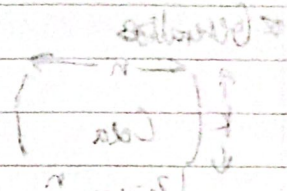
What are these 2-buckets?

If (21) is clearly the line (2, 1) then they are the same line: = 1

Any $\langle abc \rangle$ with $a, b, c \in \{1, 2, 3, 4, 5\}$

$$\text{put "2"} = (23) \cap (15) \neq 2 \langle 315 \rangle - 1 \langle 215 \rangle = 1 \langle 523 \rangle - 5 \langle 123 \rangle$$

$$\text{put "4"} = (34) \cap (15) \neq 3 \langle 415 \rangle - 4 \langle 315 \rangle = 1 \langle 534 \rangle - 5 \langle 134 \rangle$$



(12)(34) = (14)(23) = (13)(24) = (14)(23) = (13)(24) = (14)(23) = (13)(24)

(12)(34) = (14)(23) = (13)(24) = (14)(23) = (13)(24) = (14)(23) = (13)(24)

(12)(34) = (14)(23) = (13)(24) = (14)(23) = (13)(24) = (14)(23) = (13)(24)

(12)(34) = (14)(23) = (13)(24) = (14)(23) = (13)(24) = (14)(23) = (13)(24)

(12)(34) = (14)(23) = (13)(24) = (14)(23) = (13)(24) = (14)(23) = (13)(24)

(12)(34) = (14)(23) = (13)(24) = (14)(23) = (13)(24) = (14)(23) = (13)(24)

$$\langle (11) \rangle = \langle (11)(51) \rangle$$

(12)(34) = (14)(23) = (13)(24) = (14)(23) = (13)(24) = (14)(23) = (13)(24)

$$\langle (12) \rangle = \langle (12)(34) \rangle$$

$$\langle (13) \rangle = \langle (13)(24) \rangle$$

$$\langle (14) \rangle = \langle (14)(23) \rangle$$

$$\langle (15) \rangle = \langle (15)(24) \rangle$$

$$\langle (23) \rangle = \langle (23)(15) \rangle$$

quadrato, cubo, quatio, quintio, sextio, septio, octo, nono, decio, 11, dodectio

Problem set 2

#4 Just by counting parameters, any 5 points in $\mathbb{P}^2$: $X_1^I, \dots, X_5^I$ determine a conic C_{IS}

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad \leftarrow x, y, z \text{ are 3 points? or } [x:y:z]$$

$$\downarrow \text{homogenize}$$

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0 : C_{IS} X^I X^J = 0 \text{ with } C_{IS} \text{ symmetric}$$

$$S \text{ points: } \binom{5}{2} + \underset{\text{deg}}{5} = \frac{5 \cdot 4}{2} + 5 = 10 + 5 = 15$$

No: $\mathbb{P}^2$: x, y, z are 3 variables: $\binom{3}{2} + 3 = 3 + 3 = 6$ (A, B, C, D, E, F) but projective, so 5 degrees of freedom

Can determine A, B, C, D, E, F by plugging in the 5 points $X_1, \dots, X_5$ for x, y, z ?

what's the minimal set of equations to use?
+ just 6 equations in 6 unknowns

So if we are given six points, they must satisfy a single relation for all six to lie on a conic

$$S = \sum_{(I,J) \in \binom{[6]}{2}} (X_1^I X_1^J) \dots (X_6^I X_6^J) = 0$$

This is some quadratic expression in 3-variables, of weight 2 in each of the X_i

3 brackets: $\mathbb{P}^2$

quadratic by weight 2 (in 6 points)

here the 6 are

Condition for 6 points in $\mathbb{P}^2$ to lie on a plane

This is Veronese (d-uple): $(pB \text{ Schubert})$

$n, d > 0$ monomials of degree d in $2+1$ variables x_0, x_1, x_2

$$N = \binom{n+d}{d} - 1 = \binom{9}{2} - 1 = \frac{9 \cdot 8}{2} - 1 = 35$$

Embedding: $\mathbb{P}^2 \rightarrow \mathbb{P}^5$

" C_{IS} is a hyperplane in $\mathbb{P}^5$ " $\rightarrow$ Prove that $\mathcal{C}_{IS}$ as given by A, B, C, D, E, F and the equation $Ax^2 + Bxy \dots = 0$, being one equation: hence defining a codim-1 thing?

$$\mathbb{P}^5 \supseteq \begin{pmatrix} x \\ y \\ z \\ x^2 \\ y^2 \\ z^2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ a \\ b \\ c \\ d \\ e \end{pmatrix} \text{ so a hyperplane: } \sim \begin{pmatrix} 1 \\ a \\ b \\ c \\ d \\ e \end{pmatrix}$$

6 points in $\mathbb{P}^2$ lie on a hyperplane: $\langle x_1, \dots, x_6 \rangle = 0$

hype: $\langle x^2, y^2, z^2, xy, xz, yz \rangle = 0$

$$\det \begin{pmatrix} (x^2)_1 & (x^2)_2 \\ (y^2)_1 & (y^2)_2 \\ (z^2)_1 & (z^2)_2 \end{pmatrix} = 0$$

$$S = \varepsilon_{(I_1 J_1) \dots (I_6 J_6)} (x_{I_1}^{J_1} x_{I_2}^{J_2}) \dots (x_{I_6}^{J_6} x_{I_7}^{J_7}) = 0$$

This is some quadratic expression in 3-brackets, of weight 2 in each of the x_i

degree $2 \cdot 6$ terms: = degree 12 expression

must be 3 brackets: that's all we have for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

so by $\frac{12}{3}$ we have 4 of them

Why weight 2 exactly for each? It's in a determinant: there is one entry from each row and column: for example exactly along the left column, we have:

$$x_1^2(\text{---}) + y_1^2(\text{---}) + z_1^2(\text{---}) + x_1 y_1(\text{---}) + x_1 z_1(\text{---}) + (y_1 z_1)(\text{---})$$

so clearly weight 2 in t for the map: $[x_i, y_i, z_i] \mapsto [tx_i, ty_i, tz_i]$ pulls out a t^2

which means, since determinants pick out a single factor of t , we need 2 of them

$S=0$ for 6 points on a curve

Monic curve $X(t) = (1, t, t^2)$: $A(t^2 + Bt) + C(t^2 + Dt) + E(t) + F(t^2) = A + Bt + C + Dt + E + Ft^2$

? or should we think: $z=1, x=t, y=t^2$: $Ay + Dx + F = At^2 + Dt + F$

pick six: $(1, t_1, t_1^2), (1, t_2, t_2^2), \dots, (1, t_6, t_6^2)$

$$(x_a x_b x_c) = \begin{vmatrix} 1 & 1 & 1 \\ t_a & t_b & t_c \\ t_a^2 & t_b^2 & t_c^2 \end{vmatrix} = t_b t_c^2 - t_b^2 t_c + t_c t_a^2 - t_a t_c^2 + t_a t_b^2 - t_b t_a^2$$

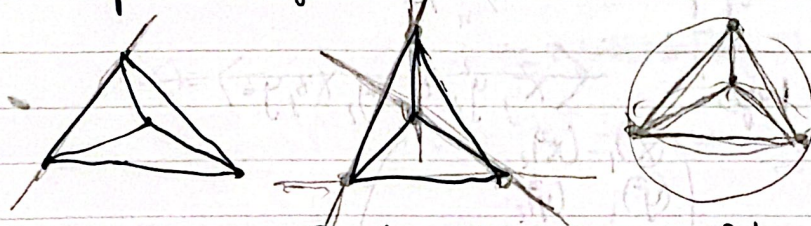
$$(abc) = (t_b - t_a)(t_c - t_b)(t_c - t_a) = (t_b t_c - t_a t_c - t_b^2 + t_a t_b)(t_c - t_a) = t_b t_c^2 - t_a t_c^2 - t_b^2 t_c + t_a t_b t_c - t_a^2 t_c + t_a^2 t_b$$

the $S=0$ on this curve by Eq (5): check (1):

First term: $(123) + (567)$: $(t_2 - t_1)(t_3 - t_1) \dots (t_7 - t_6)(t_1 - t_6) \rightarrow$ better hope minus signs work out

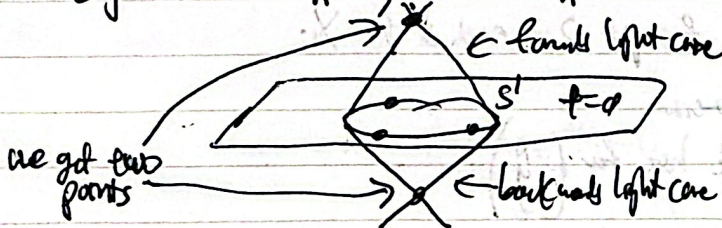
Second term: $(612) + (351)$: $(t_1 - t_6)(t_2 - t_1) \dots (t_1 - t_3)(t_1 - t_5)$

5) By 4 part in $\mathbb{R}^3$ is a sphere: $x^2 + y^2 + z^2 = r^2$



Andrew: think about those 4 parts as S^2 not in $\mathbb{R}^3$, but now in $\mathbb{R}^4 \rightarrow$ do a boost so that this S^2 lies in the same time slice at $t=a$. Then look at both the forward and backward ~~light~~ light cones of this S^2 .

E.g. instead of $\mathbb{R}^3$ let's do $\mathbb{R}^{1,2}$ with S^1 instead of S^2 :



k : # jump back
 $\#$ outgoing (or ingoing) in perfect orientation
 $\#$ it goes over $\rightarrow$

read for $Gr(K, n)$
 read $n-1$ to count energy, as all further
 BCFW's are loops, top cell is $K(n-k)$ - dim.
 so read $K(n-k)$ BCFW bridges = degree freedom
 $\Rightarrow$ top cell has $Kn - K^2 - (n-1)$ loops

Problem Set 3

1) BCFW for NMHV

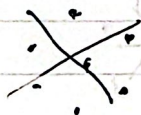
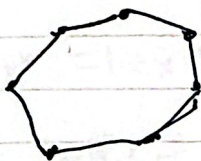
$K=1$ $k=3$

6 terms for $n=7$ ($k=1$)

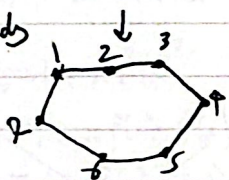
regular: $Gr(1, 7)$, momentum twistor: $Gr(3, 7)$

3 vertices in 7-D, or 7 vertices in 3D

2-D projected

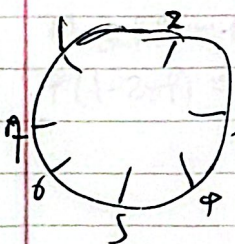
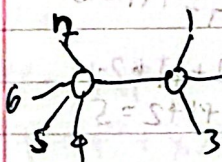


bonds

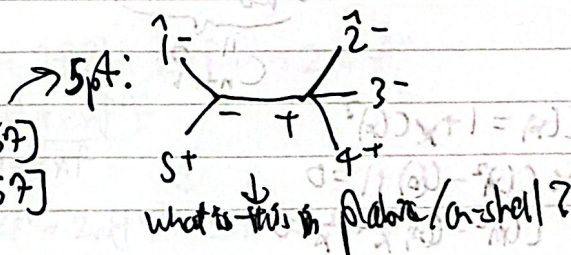


$1 \rightarrow 3$
 $2 \rightarrow 5$
 $3 \rightarrow 6$
 $4 \rightarrow 7$
 $5 \rightarrow 8$
 $6 \rightarrow 9$
 $7 \rightarrow 11$

$\sigma = (35678911)$ (87365911)
 (53678911) (126375911)
 (86978911) (87395611)
 (65378911) (89375611)
 (67358911) (89311567)
 (76358911)
 (7638911)
 (78365911)



$[12345] + [12356] + [12367]$
 $+ [13456] + [13467] + [14567]$



what is this in picture/or-shell?

$(x_1 x_2 x_3) / (x_1 x_2 x_3) + 1 = x_1 x_2 x_3 = x_1 x_2 x_3 + 1 = 0$

$(x_1 x_2 x_3 x_4 x_5 x_6 x_7) / (x_1 x_2 x_3 x_4 x_5 x_6 x_7) + 1 = x_1 x_2 x_3 x_4 x_5 x_6 x_7 + 1 = 0$

$1 = 0$

recursion relations - p312, 313

$$M_{n,k} = M_{n-1,k} + \sum [j \text{ from } 1 \text{ to } n-1] M_R(I, -, j, I_j) M_L(I, j, j+1, -, \hat{n}_j)$$

$$M_{5,3} = M_{4,3} + \sum$$

$$M_{5,3} = M_{5,3} + \sum_{j=2}^3 [2 \ 2 \ 1 \ 5 \ 6] \begin{array}{c} 6 \\ | \\ 5 \\ | \\ 4 \\ | \\ 3 \end{array} \begin{array}{c} 1 \\ | \\ 2 \end{array} \leftarrow = [1 \ 3 \ 4 \ 5 \ 6] \text{ from reduction/reunion}$$

$$\sum_{j=3}^3 [3 \ 4 \ 5 \ 6 \ 1] \begin{array}{c} 6 \\ | \\ 5 \\ | \\ 4 \\ | \\ 3 \end{array} \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 3 \end{array}$$

$$= [1 \ 2 \ 3 \ 4 \ 5] + [1 \ 2 \ 3 \ 5 \ 6] [1 \ 3 \ 4 \ 5 \ 6]$$

$$2) C(x) = \sum_n C_n x^n$$

$$C(x) = 1 + x C(x)^2$$

$$1 + x C(x)^2 = 1 + x \left(\sum_n C_n x^n \right)^2$$

$$= 1 + x \left(\sum_i C_i x^i \right) \left(\sum_j C_j x^j \right)$$

$$= 1 + x \sum_{n=0}^{\infty} \sum_{k=0}^n C_k C_{n-k} x^n$$

$$= 1 + \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^n C_k C_{n-k}}_{C_{n+1}} x^{n+1}$$

$$C_0 = 1$$

$$C_1 = \sum_{j=0}^1 C_j C_{1-j} = C_0 C_1 + C_1 C_0$$

$$C_1 = 1$$

$$C_2 = \sum_{j=0}^2 C_j C_{2-j} = C_0 C_2 + C_1 C_1 + C_2 C_0$$

$$C_3 = \sum_{j=0}^3 C_j C_{3-j} = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0 = 1 + 1 + 2 = 2$$

$$C_4 = C_1 C_3 + C_2 C_2 + C_3 C_1 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 2 + 1 + 2 = 5$$

$$C_5 = C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1$$

$$= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1$$

$$= 5 + 2 + 2 + 5 = 14$$

$$C(x) = 1 + x C(x)^2$$

$$x C(x)^2 - C(x) + 1 = 0$$

$$C(x)^2 - C(x) \frac{1}{x} + \frac{1}{x} = 0$$

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} \frac{(2n)!}{n! n!}$$

$$C = 1 + x C^2 = \sum_n C_n x^n = 1 + x \left(\sum_i C_i x^i \right) \left(\sum_j C_j x^j \right)$$

$$1 + C_1 x^1 + C_2 x^2 + C_3 x^3 = 1 + x (1 + C_1 x^1 + C_2 x^2 + \dots) (1 + C_1 x^1 + C_2 x^2 + C_3 x^3 + \dots)$$

$$C_1 = 1$$

$$C_2 =$$

$$C = 1 + xC^2$$

$$xC^2 - C + 1 = 0$$

$$0 = xC^2 + (-1)C$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(x)(1)}}{2x} = \frac{1 \pm \sqrt{1 - 4x}}{2x} = \frac{1}{2x} \pm \frac{\sqrt{1 - 4x}}{2x}$$

$$\sum_n C_n x^n = 1 + x \sum_{i,j} C_i x^i C_j x^j$$

$$\sum_n C_n x^n = 1 + x \sum_{i,j} C_i C_j x^{i+j}$$

$$C_n = \text{coeff of } i+j=n-1$$

$$f(x) = 1 + x f(x)^2 \rightarrow f(x)^2 - \frac{f(x)}{x} + 1 = 0 : x f(x)^2 - f(x) = -1$$

$$0 = 1 + x f(x) - f(x)$$

$$0 = 1 + f(x)(x-1)$$

$$f(x)(x-1) = -1$$

$$f(x) = -\frac{1}{x-1} = \frac{1}{1-x}$$

$$\text{if } |x| < 1$$

$$f'(x) = -\frac{1}{(1-x)^2} (-f - x f')$$

$$f(x)(x f(x) - 1) = -1$$

$$f(x)(1 - x f(x)) = 1$$

$$\frac{1}{f(x)} = 1 - x f(x)$$

$$f(x) = \frac{1}{1 - x f(x)}$$

or

$$f = 1 + x f^2$$

$$f' = f^2 + x 2 f f'$$

$$f' - 2 x f f' = f^2$$

$$f'(1 - 2 x f) = f^2 \rightarrow f' = \frac{f^2}{1 - 2 x f}$$

$$f'' = 2 f f' + 2 f f' + 2 x f' f' + 2 x f f''$$

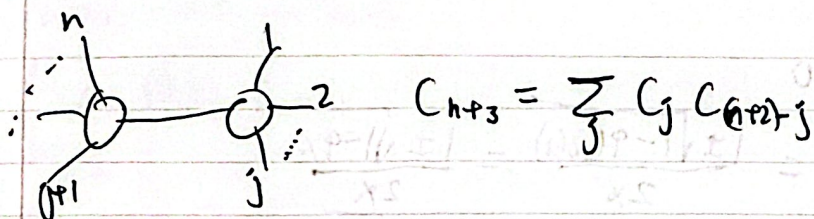
$$f'' = \frac{4 f f' + 2 x f'^2}{1 - 2 x f}$$

$$4 f f' + 2 x f'^2 + 2 x f f''$$

$$f''' = \frac{4 f' f' + 4 f f'' + 2 f f' + 2 x f' f'' + 2 f f''}{1 - 2 x f}$$

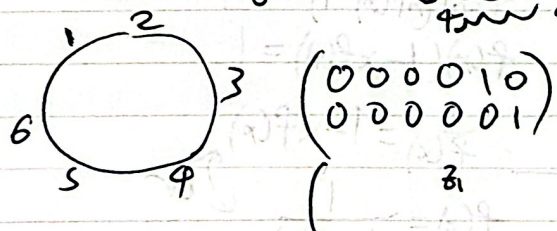
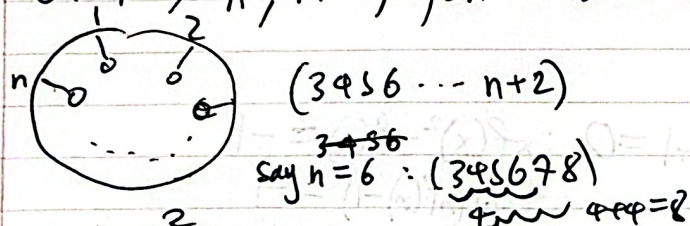
$$= \frac{6 f' f' + 6 f f'' + 6 x f' f''}{1 - 2 x f}$$

$$+ 2 x f f'' + 2 x f f'''$$



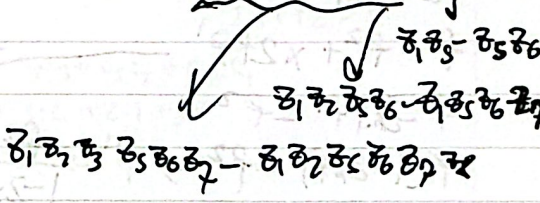
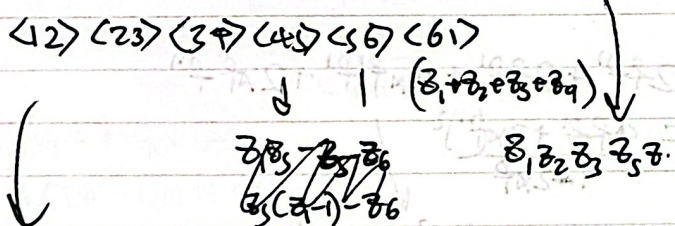
$$C_0=1, C_1=1, C_2=2, C_5=5$$

④ $G+(Z, n), z_A, A=1, \dots, 2n-4$ for the top-cell of $G+(Z, n)$



$$\begin{pmatrix} 0 & z_1+z_2+z_3+z_4 & z_1+z_2+z_3 & z_1+z_2 & z_1 & 1 & 0 \\ 1 & 0 & z_1+z_2+z_3 & z_1+z_2 & z_1 & z_1 & 1 \end{pmatrix} = \begin{pmatrix} z_1 z_2 z_3 z_4 & z_1 z_2 z_3 & z_1 z_2 & z_1 & 1 & 0 \\ 0 & z_1 z_2 z_3 & z_1 z_2 & z_1 & z_1 & 1 \end{pmatrix}$$

$$C_{n+1} = C_n + \phi C_n$$



$$[12345678] [123567 - 125678] [1256 - 1567] [15 - 56] \cdot 1 [1234]$$

$$\begin{pmatrix} z_1 z_2 z_3 & z_1 z_2 & z_1 & 1 & 0 \\ 0 & z_1 z_2 & z_1 & z_1 & 1 \end{pmatrix} \begin{pmatrix} (3-8)(2-7)(1-6) \\ 1=1 & 2 & 3 & 4 & 5 & 6^{th} \text{ min} \\ 0 & 3 & 2 & 1 & 1 & 0 \end{pmatrix}$$

$$[123456] [1245 - 1456] [14 - 45] \cdot 1 [123] \begin{pmatrix} z_1 z_2 z_3 & z_1 z_2 & z_1 & 1 & 0 \\ 0 & z_1 z_2 & z_1 & z_1 & 1 \end{pmatrix} \begin{pmatrix} (2-6)(1-5) \\ 1=1 & 2 & 3 & 4 & 5 & 6^{th} \text{ min} \\ 0 & 3 & 2 & 1 & 1 & 0 \end{pmatrix}$$

$$z_{n-1+1} - z_{n+1+1} + (n-1) = z_{n-1+1} - z_{2n-1}$$

$$z_2 \rightarrow z_2 - z_6 \quad z_1 \rightarrow z_1 - z_5$$

$$z_6 \rightarrow z_6 - z_5$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 z_2 z_3 & 1 & 0 \\ 0 & z_3 z_4 & 1 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} a z_1 z_2 & a z_1 + b z_3 z_4 & a + b z_3 \\ c z_1 z_2 & c z_1 + d z_3 z_4 & c + d z_3 \end{pmatrix}$$

$$(a c z_1^2 z_2 + a d z_1 z_2 z_3 z_4 - a c z_1^2 z_2 - b c z_1 z_2 z_3 z_4) = (ad - bc)(z_1 z_2 z_3 z_4)$$

$$\begin{aligned} & \cdot (a c z_1 + b c z_3 z_4 + a d z_1 z_3 + b d z_3^2 z_4 - a c z_1 - a d z_3 z_4 - b c z_1 z_3 - b d z_3^2 z_4) \\ & = (bc - ad)(z_3 z_4) + (ad - bc) z_1 z_3 \\ & = (ad - bc)(z_1 z_3 - z_3 z_4) \\ & = z_3 (ad - bc)(z_1 - z_4) \end{aligned}$$

$$\cdot (ad + b d z_3 - bc - b d z_3) = (ad - bc)$$

$$\cdot (-1)(bc z_1 z_2 - ad z_1 z_2) = (ad - bc)(z_1 z_2)$$

$$(ad - bc)^4 [1234] [12] [1-4]$$

$$\begin{pmatrix} z_1(z_1+z_2+z_3+z_4) & z_2(z_2+z_3+z_4) & z_3(z_3+z_4) & z_4 z_4 & 0 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix}$$

$$z_1 z_4 (z_1 + z_2 + z_3 + z_4) - z_2 z_4 (z_2 + z_3 + z_4) = z_1 z_4 z_5$$

$$z_1 z_4 (z_2 + z_3 + z_4 - z_4) = z_1 z_4 z_5$$

$$z_1 z_4 z_5$$

$$z_1 z_4 z_5$$

$$\begin{aligned} & z_1^2 z_2 z_3 z_4 z_5^2 z_6^2 z_7^2 z_8^2 (z_1 + z_2 + z_3 + z_4) \\ & = z_1^2 z_2 z_3 z_4 z_5^2 z_6^2 z_7^2 z_8^2 \\ & + z_1^2 z_2^2 z_3^2 z_4^2 \end{aligned}$$

4) Solution:

Think: top cell in $G(2, n)$ has permutation: $\sigma = (3 \ 4 \ 5 \ \dots \ n \ n+1 \ n+2)$

It's clear that going down in d to a point involves $(n-2)$ branches for $n+1$, and $(n-2)$ for $n+2$:

So we get the representative matrix as,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & z_1 z_2 z_3 \dots z_{n-2} & \dots & z_1 z_2 z_3 \dots z_{n-2} \end{pmatrix}$$

$$\hat{C}_{a+1} = C_{a+1} + \omega C_a$$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & z_1 z_2 z_3 \dots z_{n-2} & \dots & z_1 z_2 z_3 \dots z_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & y_1 & 0 & \dots & 0 \\ 0 & 1 & z_1 & z_2 z_3 \dots z_{n-2} & \dots & z_1 z_2 z_3 \dots z_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & y_1 + y_2 y_3 & \dots & y_1 y_2 y_3 \\ 0 & 1 & z_1 z_2 z_3 + y_2 z_1 z_3 & \dots & z_1 z_2 z_3 + y_2 z_1 z_3 \end{pmatrix}$$

$$-1 \text{ for } (j, 30) \text{ p 60 Gramian: } C_{a+1} \rightarrow C_{a+1} + (-1)^q \omega C_a$$

$$\text{Let: } -y_1 (z_1 z_2 + y_2 z_1) - (-y_2 y_3) z_1$$

$$= -y_1 z_1 z_2 - y_1 y_2 z_1 + y_2 y_3 z_1$$

$$= y_1 z_1 z_2$$

Bole: $\begin{pmatrix} 1 & 0 & -z_{n-1} & -z_{n-1} z_n & -z_{n-1} z_n z_{n+1} & \dots & -z_{n-1} z_n z_{n+1} z_{n+2} \\ 0 & 1 & z_1 & -z_1 z_2 + z_n(z_1) & z_1 z_2 z_3 + z_{n+1}(z_1 z_2 + z_1 z_3) & \dots & z_1 z_2 z_3 + z_{n+2}(z_1 z_2 z_3 + z_1 z_2 z_{n+1} + z_1 z_3 z_{n+1}) \end{pmatrix}$

negatives are diagonal

consecutive minus
or

$$1 \cdot z_{n-1} \cdot z_1 z_2 z_{n-1} \cdot z_1 z_2 z_3 z_{n-1} z_n \cdot z_1 z_2 z_3 z_4 z_{n-1} z_n z_{n+1} \dots z_1 z_2 \dots z_{2n-1} z_{n+1} z_{n+2}$$

$$= (z_1)^{n-2} \cdot (z_2 z_{n-1}) (z_2 z_3 z_{n+1} z_1) (z_2 z_3 z_4 z_{n-1} z_n z_{n+1}) \dots (z_2 z_3 \dots z_n)$$

$$(n-1) \text{ minus } n (z_{n-1} z_1 z_{n+1} \dots)$$

Signs/Ordering is wrong: just do examples: next page

My original example: $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ ← BCFW bridge

$\downarrow$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & Id & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Id & Id \end{pmatrix} \quad \begin{matrix} Id=1 \\ I:=3, \\ \text{BCFW} \end{matrix}$$

$\downarrow$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & Id & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Id & Id \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & Id & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Id & Id \end{pmatrix}$$

so $\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle = (12345678) (12345678) (12345678) (12345678) (12345678)$
→ look 1 on each and as we move right

Andrew's example:

$$\begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_3 & x_4 & x_5 & x_6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_3 & x_4 & x_5 & x_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}$$

$$\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle = 1 \cdot (-x_1) \cdot (x_1 x_3 x_4 - x_2 x_3 x_4) \cdot (-x_3 x_4)$$

$$\text{so } \frac{1}{\langle 12 \rangle \dots \langle n1 \rangle} \prod d^{\text{dim}} C = \frac{d x_1 d x_3 (x_3 d x_4 + x_4 d x_3) (x_1 d x_2 + x_2 d x_1 + x_1 d x_4 + x_4 d x_1)}{x_1^2 x_2 x_3^2 x_4} = \frac{d x_1 d x_2 d x_3 d x_4}{x_1 x_2 x_3 x_4} = d \log x_1 d \log x_2 d \log x_3 d \log x_4$$

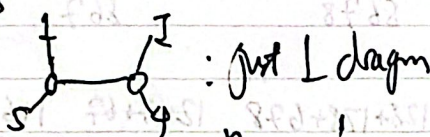
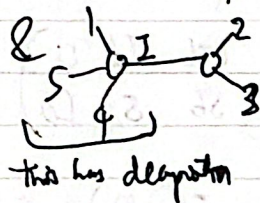
Andrew's matrix: $\begin{pmatrix} 1 & 0 & -a & -c - \frac{ad}{b} & -e & e(c + \frac{ad}{b}) \\ 0 & 1 & b & d & e & d \end{pmatrix}$

We got this by: $\begin{pmatrix} 1 & 0 & -a & -c \\ 0 & 1 & b & d \end{pmatrix}$ this way all minors are positive for $a, b, c, d > 0$ except possibly $\langle 34 \rangle = bc - ad$
 (How to get $bc - ad > 0$? $\Rightarrow c > \frac{ad}{b} \Rightarrow c - \frac{ad}{b} > 0$ so let $\tilde{c} = c - \frac{ad}{b}$
 $\begin{pmatrix} 1 & 0 & -a & \tilde{c} \\ 0 & 1 & b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & -a & -c + \frac{ad}{b} \\ 0 & 1 & b & d \end{pmatrix}$ Thmk: want c in $\tilde{c}$ to be bigger than $\frac{ad}{b}$. Since $c > 0$, add $\frac{ad}{b}$ to c , which is guaranteed to be bigger than $\frac{ad}{b}$: $-(c + \frac{ad}{b})$, Done!

Equivalently, Andrew's: $\frac{d^{\text{dim}} C}{\langle 12 \rangle \dots \langle n1 \rangle} = \frac{d a d b d c d d}{d \cdot d \cdot (bc - ad)} = \frac{d a}{d} \frac{d b}{d} \frac{d c}{d} \frac{d d}{d} = \frac{d a}{d} \frac{d b}{d} \frac{d c}{d} \frac{d d}{d}$
 $\begin{pmatrix} 1 & 0 & -a & -c \\ 0 & 1 & b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & -a & -c - \frac{ad}{b} \\ 0 & 1 & b & d \end{pmatrix}$

Solution:

2) Show that for any n , the total number of terms (for all k) contributing to BCFW at tree level are given by the Catalan numbers which satisfy $C_{n+1} = \sum_j C_j C_{n-j}$



so # diagrams for n is $\sum_{k=2}^{n-2}$

$$T^n = \sum_{k=2}^{n-2} T^{n-k+1} T^{k+1}$$

each one has the same set of diagrams

$$T^{k+1} = \sum_{j=2}^{k-1} T^{k-j+1} T^{j+1}$$

$$\sum_{k=0}^{n-4} T^{n-(k+2)+1} T^{(k+2)+1}$$

$$T^n = \sum_{k=0}^{n-4} T^{n-k-1} T^{k+3} \approx \sum_{k=0}^{n-4} T^{(n-4)-k} T^k$$

$T^3 = 1$

$C_0 = 1$

we: $T^n = C_{n+3}$, $C_{n+1} = \sum_{j=0}^n C_j C_{n-j}$

$C_{n+3} = \sum_{j=0}^{n+2} C_j C_{n+2-j}$ or $C_{n+3} = \sum_{j=0}^n C_j C_{n+2-j}$

Show that $C(x) = \sum_n C_n x^n$ satisfies $C(x) = 1 + x C(x)^2$:

$$1 + x C(x)^2 = 1 + x \left(\sum_i C_i x^i \right) \left(\sum_j C_j x^j \right) = 1 + x \sum_{i,j=0}^{\infty} C_i C_j x^{i+j}$$

$$= 1 + x \sum_{n=0}^{\infty} \sum_{k=0}^n C_k C_{n-k} x^n$$

$$= 1 + \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^n C_k C_{n-k}}_{\text{add this is } C_{n+1}, \text{ the coefficient of } x^{n+1}} x^{n+1} = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} C_k C_{n-k} x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} C_k C_{n-k} x^n$$

$$\Rightarrow C(x) = C(x)$$

$$\text{define } \sum_{k=0}^{n-1} C_k C_{n-k} = 1$$

Solve for $C(x)$:

$$1 + x C(x)^2 = C(x) \Rightarrow x C^2 - C + 1 = 0$$

Quadratic: $\frac{1 \pm \sqrt{1-4x}}{2x} \rightarrow \text{must be } - \text{ when } x=0 : \frac{1 - \sqrt{1-4x}}{2x}$

$$\sqrt{1-4x} = \frac{1}{2\sqrt{1-4x}} (-4) = -\frac{2}{\sqrt{1-4x}} \quad (\sqrt{1-4x})'' = \frac{1}{2} \cdot \frac{2}{(\sqrt{1-4x})^{3/2}} (-4) = -4(1-4x)^{-3/2}$$

$$f''' = 6(1-4x)^{-5/2} \cdot (-4) = -24(1-4x)^{-5/2}$$

$$f^{(4)} = 60(1-4x)^{-7/2} \cdot (-4) = -240$$

so for original $x=0$:

$$\sqrt{1-4x} \approx 1 - 2x +$$

Start w/ 2.

$$f^{(n)} = \frac{(2n+1)!}{2^n} (4^n) = \frac{(2n+1)!}{2^n} 2^{n+1}$$

$$n=0: 2$$

$$n=1: 3 \cdot 2 \cdot 4 = 30$$

$$n=2: 5!$$

$$2 \cdot \frac{2n!}{2^{n+1}} \cdot 4^{n+1} = \frac{(2n+1)!}{2^{n+1}} \cdot 2^{n+1} = (2n+1)(2n-1)(2n-3) \dots 1 \cdot 2^{n+1} \cdot 2$$

$$\frac{3!}{2^2} \cdot 4^2 = \frac{6}{4} \cdot 16 = 24$$

$$1 - \frac{1 - 2x - 2 \sum_{n=0}^{\infty} \frac{(2n+1)(2n-1) \dots 1 \cdot 2^{n+1}}{(n+2)!} x^{n+2}}{2x}$$

$$= \frac{2x + 2 \sum_{n=0}^{\infty} \frac{(2n+1)(2n-1) \dots 1 \cdot 2^{n+1}}{(n+2)!} x^{n+2}}{2x} = 1 + \sum_{n=0}^{\infty} \frac{(2n+1)(2n-1) \dots 1 \cdot 2^{n+1}}{(n+2)!} x^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(2n-1)(2n-3) \dots 1 \cdot 2^n}{(n+1)!} x^n = \sum_{n=1}^{\infty} \frac{1 \cdot 2(2n-1) \cdot 2(2n-3) \cdot 2 \dots 1}{(n+1)!}$$

$$\left(\frac{1}{n+1} \right) \binom{2n}{n} = \frac{1}{n+1} \frac{(2n)!}{n! n!}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{1}{n!} \frac{2n}{n} (2n-1) \frac{2(n-1)}{n-1} (2n-3) \frac{2(n-2)}{n-2} \dots \frac{2}{2} \cdot 1$$

$$= \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{1}{n!} \frac{(2n)!}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n+1} \binom{2n}{n}$$