What is a leading singularity? 10 minutes left

"Leading singularity is one of the most abused terms in physics. Ask 5 people what it means and you'll get 5 different answers"

Forewarned: It's very technical → can't do much in 30 min

So what's my goal / why do we care

If you've never seen it before, if you read a physics paper, some idea of what it means, connection

flavor!

slow, basic

My interest: connect plabic diagrams & positroid varieties to physics

confusing morass of terminology: leading singularity
unit leading singularity integrand
maximal cut
generalized unitarity

I'll try to make some sense of these

Can get quite technical:

Felt yesterday while preparing the talk that this isn't a great topic for mathematicians → really for a new physics graduate starting out in the field.

If I make mistakes, Nima is free to jump in

In particular:

(1) Definition of leading singularity
(A)

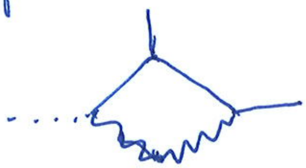
(2) Why we care / connection to amplitude
relationship

(3) Where diagrams show up

This talk: $N=4$ SYM (simplest theory — otherwise too complicated)
(1) fewer (2) particles massless (3) $D=4$ spacetime dimensions
everything will be a bit simplified

Definition:
A Leading singularity is a solution to a maximal cut:

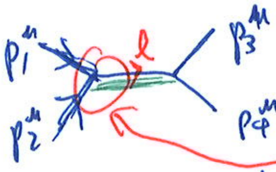
An amplitude consists of diagrams like

$\sum$  $\xrightarrow{\text{rules}}$ Algebraic Expression

sum

Let's be a bit more specific:

remember simplification that all particles massless



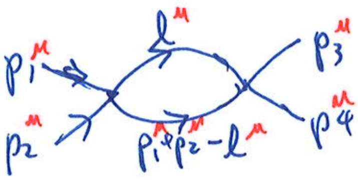
$\xrightarrow{\text{rules}} \dfrac{1}{(l-m)^2} = \dfrac{1}{l^2} = \dfrac{1}{(p_1+p_2)^2}$

momentum conservation at each vertex

internal lines show up squared in the denominator "propagator"

Schwartz talked about these Schwinger parameters

Specify the momentum 4-vectors on the outside



$\xrightarrow{\text{rules}} \displaystyle\int \frac{dl}{l^2 (p_1+p_2-l)^2}$

l is undetermined: physics says that we need to sum over all possibilities = take integral

Now, l is a vector: $\vec{l} = (l_0, l_1, \dots, l_D)$, $D = 3$

so we actually have: $\iiiint d^4 l$

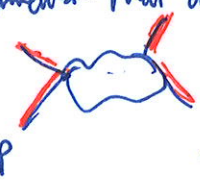
NAM
tech engineer
postdoc
what would you recommend
Python JAX

We can now say what a "cut" is:

Imagine that I take

What does this do on the $\xrightarrow{\text{rules}}$ algebraic side?

Well: remember that all external lines:



are real particles: we measure them with a detector, so they are all on mass-shell:

$p^2 = m^2$

The internal propagators are so-called "virtual particles":

we can't measure them

they don't satisfy $p^2 = m^2$

BUT:  eliminate this propagator, turning it into a real particle:

$$\int \frac{d^4 l}{l^2} \cdots \xrightarrow[\substack{\text{propagator} \\ \text{(no longer an internal line)}}]{\text{eliminate}} \int d^4 l \cdots \xrightarrow[\substack{\text{(not virtual)} \\ \text{so } l^2 = 0}]{\text{it's real now}} \int d^4 l \cdots \delta_{+}(l^2)$$

δ (lecture) $\delta^{(+)}(p^2) = \theta(E_p)\,\delta(p^2)$

zeroth component

light cone

$\downarrow$

$\int d^3 l \cdots \delta(l^2)$ since $\delta(l^2)$ imposes a condition on l that drops the dimensions to be integrated by 1

$\downarrow$

Jacobian

$$\boxed{J \sum_{\text{internal states}} \int d^3 l \cdots \delta(l^2)}$$

because $+\,-$ or $-\,+$ helicity or other information that now you measure (everything about the state)

I would like to ignore this

Let's rewrite this in a suggestive way: (suppose that l is just a number)

$$\int_{l=-\infty}^{\infty} \frac{d^4 l}{l^2} f(\dots) \to \cancel{\int d^4 l}\, f(\dots)\,\delta(l^2)$$

$$\text{“=”} \lim_{l^2 \to 0} f(\dots)$$

not sure if any of this is rigorous

$$\text{“=”} \operatorname*{Res}_{l^2=0} \frac{f(\dots)}{l^2} \qquad \left(= \lim_{l^2\to 0} (l^2 - 0) \frac{f(\dots)}{l^2} \right)$$

$$\text{“=”} \left(\frac{1}{2\pi i}\right) \oint_\gamma \frac{f(\dots)}{l^2}\, dl$$

$\Rightarrow$ so taking the cut is "like"

$$\int_{-\infty}^{\infty} \frac{dl}{l^2} f(\dots) \longmapsto \oint_\gamma \frac{dl}{l^2} f(\dots)$$

complex

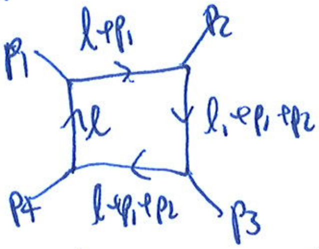
contour surrounding pole

easy to calculate: $= \operatorname*{Res}_{l^2=0} \frac{f(\dots)}{l^2}$
(integrating hard)

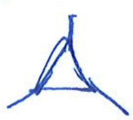
2 observations

There's a number of subtleties I skipped / don't know, but I will say:

① Since we had $\int d^4 l \dots$ to get rid of all integrations, we need to cut 4 propagators:
so we need something like

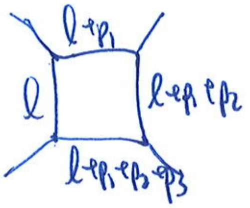


We can cut a maximum of 4 propagators in a loop (otherwise we would have leftover $\delta(\dots)$ functions / be overconstrained)

"Maximal cut" means we cut all propagators,
so  $\rightarrow$  is also a maximal cut, but it has a left over integration

② Second issue:
which pole? $\text{Res}_{l^2=0}$ isn't really a thing

If we take



and we cut all of them,

we set $l^2 = (l+p_1)^2 = (l+p_1+p_2)^2 = (l+p_1+p_2+p_3)^2 = 0$ (with the $\delta(l^2)\cdot\delta((l+p_1)^2)\cdot\delta((l+p_1+p_2)^2)\,\delta((l+p_1+p_2+p_3)^2)$)

which has 2 solutions since these are quadratic

In general, for L loops, there are 2^L solutions (e.g.  has 4 solutions etc.)

Each solution is a <u>leading singularity</u>

To recap: A leading singularity is a choice of contour (or pole) in the maximal cut:

when you go from $\int_{-\infty}^{\infty} \frac{d^4l\, f(\cdots)}{(l^2)(\cdots)} \to \oint_\gamma d^4l\, f(\cdots)$

It's (basically) the same thing, but the way it is calculated in practice is that $f(\cdots)$ comes from the <u>rules</u> for

all the ~~pieces~~ tree-level pieces

(after cutting). You solve $l^2 = \cdots = (l+p_1+p_2+p_3)^2 = 0$ for l, get one of the solutions, and plug into $f(\cdots)$ (and sum over states $\sum_{\text{internal}}$)

Note:

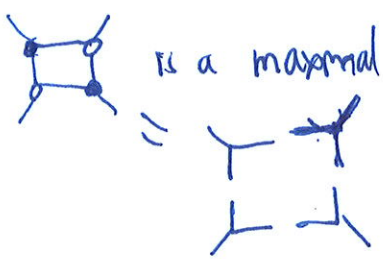
If you ~~have~~ a diagram entirely with trivalent vertices, you can think of cutting all edges, and then each $\genfrac{}{}{0pt}{}{a}{b}\!\succ c$ either is MHV or $\overline{\text{MHV}}$

(Simpler: $\begin{pmatrix}\lambda_a^1 & \lambda_b^1 & \lambda_c^1 \\ \lambda_a^2 & \lambda_b^2 & \lambda_c^2\end{pmatrix} \cdot \begin{pmatrix}\tilde\lambda_a^1 & \tilde\lambda_b^1 & \tilde\lambda_c^1 \\ \tilde\lambda_a^2 & \tilde\lambda_b^2 & \tilde\lambda_c^2\end{pmatrix} = 0$ by momentum conservation

$\Rightarrow$ λ and $\tilde\lambda$ are (2) orthogonal 2-planes in 3-dimensions, so one of them is degenerate: either the (λ) matrix or the ($\tilde\lambda$) matrix has rank 1

so each $\succ$ must be either: $\bullet$ or $\circ$

Thus



is a maximal cut diagram (with all edges on-shell)

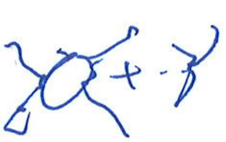
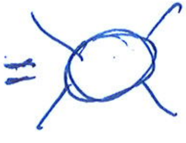
That's why it's called an "on-shell diagram" (= plabic diagram)

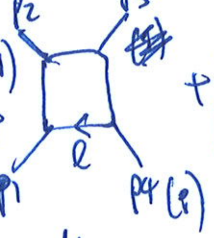
where the choice of which $\succ$ are $\bullet$ or $\circ$ corresponds to which solution/pole/contour (exact process ~~not known~~ I don't understand: change to momentum twistor space, etc)

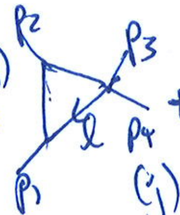
Why do we care/relationship to amplitude:

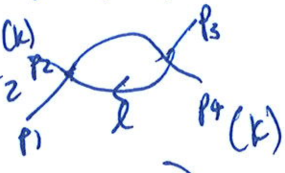
Generalized unitarity:

For 1-loop, can write an ansatz for expanding the amplitude in a basis of scalar integrals:

$\sum_{\text{all diagrams}}$  $+ \dots =$  $= \sum_i C_D^{(i)} I_D^{(i)} + \sum_j C_{D-1}^{(j)} I_{D-1}^{(j)} + \dots + \sum_k C_2^{(k)} I_2^{(k)}$ + rational terms

$= \sum_i C_4^{(i)}$ 

$+ \sum_j C_3^{(j)}$ 

$+ \sum_k C_2^{(k)}$ 

$$\int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (l-p_1)^2 (l-p_1-p_2)^2 (l+p_4)^2} \qquad \int \frac{d^D l}{(2\pi)^4} \frac{1}{l^2 (l-p_1)^2 (l-p_1-p_2)^2}$$

These drop out in planar N=4

$$\int \frac{d^D l}{(2\pi)^4} \frac{1}{l^2 (l-p_1-p_2)^2}$$

+ ~~rational~~ terms

~~apply maximal cuts both sides~~

These "scalar integrals" are easier to do the integral: so to get the amplitude, you just need to calculate the coefficients $C_D^{(i)}$

apply maximal cuts to both sides

Left side: get  leading singularities

Sum over the leading singularities

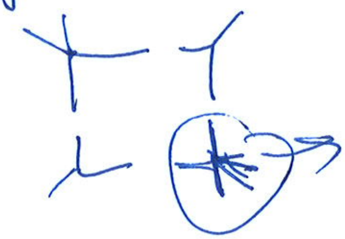
Right side:

Fact - in planar N=4 SYM, can choose these integrals on the right to be unit leading singularity integrals, which means the maximal cut = 0, ±1,

$\Longrightarrow$ coefficient C_4 = leading singularity

(only the box diagrams arise in the calculation in N=4, but in other theories, the quadruple cut $\delta^1 \delta^2 \delta^3 \delta^4$ kills the triangle and bubble diagrams, so you can get C_4. Then you can apply a triple cut to get C_3, etc. to get C_2 $\Rightarrow$ the C_3 and C_2 will be linear combinations of leading singularities)

If you have a box like:



these pieces can be computed with another notion of leading singularity:

These are computed at tree-level recursively with a procedure called BCFW.

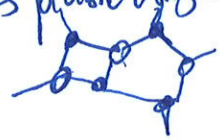
The individual terms in BCFW for planar $N=4$ SYM can be computed with the "twistor formula":

$$\tilde{\mathcal{L}}_{n,k}(\mathcal{W}_A) = \int \frac{d^{n\times k} C_{ai}}{GL(k)\prod_{j=1}^{n} M_j} \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{\ell=1}^{n} C_{a\ell}\mathcal{W}_\ell^A\right)$$

$$= \oint_{C\subset\Gamma_\sigma} \frac{d^{k\times n} C}{\mathrm{vol}(GL(k))} \frac{\delta^{k\times 4}(C\cdot\tilde{\eta})}{(1\cdots k)\cdots(n\cdots k-1)} \delta^{k\times 2}(C\cdot\tilde{\lambda})\, \delta^{2\times(n-k)}(\lambda\cdot C^{\perp})$$

mom cons $\quad$ λ-plane $\subseteq C^{\perp}$

generates all Yangian invariants/
produces all leading singularities in planar $N=4$

Each C is a parameterization of a positroid variety $\Longleftrightarrow$ planar diagram  $\Longleftrightarrow$ choice of pole/residue/contour in BCFW