Part 2 of Grassmannian Integrals:

Grassmannian for scattering amplitudes in 4d $\mathcal{N} = 4$ SYM and 3d ABJM by Elvang, Huang, Keeler, Lam, Olson, Roland, and Speyer

Joseph Fluegemann

Cornell University

Geometry and Amplitudes Journal Club, Fall 2023

Review from Last Time

For n-particle N^{k}MHV amplitudes in 4d planar $\mathcal{N}=4$ SYM there are three formulations of the kinematic data:

twistor space		momentum space		momentum twistor space
$\mathcal{W}_i = (\tilde{\mu}_i, \tilde{\lambda}_i \,\vert\, \tilde{\eta}_i)$	$\underset{\substack{\text{Fourier transform} \\ \text{eq (2.6)}}}{\longleftrightarrow}$	$(\lambda_i, \tilde{\lambda}_i \,\vert\, \tilde{\eta}_i)$	$\underset{\substack{\text{incidence relations} \\ \text{eq (2.10)}}}{\longleftrightarrow}$	$\mathcal{Z}_i = (\lambda_i, \mu_i \,\vert\, \eta_i)$

One important thing is that whereas the first two (twistor and momentum) spaces have corresponding Grassmannians $Gr(k+2, n)$, momentum twistor space corresponds to the Grassmannian $Gr(k, n)$ of 2 dimensions less (we saw this happens because the amplitude has the MHV amplitude factored out).

Review: Important formulas from last time

Momentum Conservation: $\sum_{i=1}^{n} |i\rangle [i| = 0$

Plucker/Schouten: $\langle x, y\rangle |z\rangle + \langle z, x\rangle |y\rangle + \langle y, z\rangle |x\rangle = 0$. Note: these are all 2D vectors. Short Exercise: Prove.

Review: Important formulas from last time

Momentum Conservation: $\sum_{i=1}^{n} |i\rangle [i| = 0$

Plucker/Schouten: $\langle x, y\rangle |z\rangle + \langle z, x\rangle |y\rangle + \langle y, z\rangle |x\rangle = 0$. Note: these are all 2D vectors. Short Exercise: Prove.

One answer: Since three 2D vectors must be linearly dependent, we can write: $a|z\rangle + b|y\rangle + c|x\rangle = 0$.

If we contract with $\langle z|$, we get: $a\langle zz\rangle + b\langle zy\rangle + c\langle zx\rangle = 0$, so $b = -\frac{\langle zx\rangle}{\langle zy\rangle}c$.

If we contract with $\langle y|$, we get: $a\langle yz\rangle + b\langle yy\rangle + c\langle yx\rangle = 0$, so $a = -\frac{\langle yx\rangle}{\langle yz\rangle}c$.

Thus, $0 = a|z\rangle + b|y\rangle + c|x\rangle = -\frac{\langle yx\rangle}{\langle yz\rangle}c|z\rangle - \frac{\langle zx\rangle}{\langle zy\rangle}c|y\rangle + c|x\rangle$.

Multiplying everything by $\frac{\langle yz\rangle}{c}$, we get: $0 = -\langle yx\rangle |z\rangle + \langle zx\rangle |y\rangle + \langle yz\rangle |x\rangle$

Dual space is defined by the relation: $p_i^\mu = y_i^\mu - y_{i+1}^\mu$

Then we get the momentum twistors $Z_i^I = (|i\rangle, [\mu_i|)$ defined by $[\mu_i|^a := \langle i|_{\dot{a}}\, y_i^{\dot{a}a} = \langle i|_{\dot{a}}\, y_{i+1}^{\dot{a}a}$

Correction/Exercise

Last time, we saw that a point Z_i^I corresponds to a null-line in y-space (the null line passing through y_i and y_{i+1}), and conversely, a line in Z-space determines a point in y-space. We proved this via the formula:

$$y_i^{\dot{a}a} \;=\; \frac{|i\rangle^{\dot{a}}\,[\mu_{i-1}|^{a} - |i-1\rangle^{\dot{a}}\,[\mu_i|^{a}}{\langle i-1, i\rangle}\,.$$

The proof of this formula was:

$$|i\rangle^{\dot{b}}\,[\mu_{i-1}|^{a} - |i-1\rangle^{\dot{b}}\,[\mu_i|^{a} \;=\; \Big(|i\rangle^{\dot{b}}\langle i-1|_{\dot{a}} - |i-1\rangle^{\dot{b}}\langle i|_{\dot{a}}\Big)\, y_i^{\dot{a}a} \;=\; \langle i-1, i\rangle\, y_i^{\dot{b}a}$$

I incorrectly said that this formula had an error. Exercise: show that the last equality in this proof is correct.

Correction/Exercise

Answer: They key thing is to be very careful with signs and raising/lowering via the epsilon symbol (in particular, which index on the epsilon symbol matters).
For this, the main formulas are:

$\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{ab} = -\epsilon_{\dot{a}\dot{b}}$, which gives us:
$|p\rangle^{\dot{a}} = \epsilon^{\dot{a}\dot{b}} \langle p|_{\dot{b}}$ and $\langle p|_{\dot{a}} = \epsilon_{\dot{a}\dot{b}} |p\rangle^{\dot{b}}$.

We first rewrite everything in kets. On the right:
$\langle i-1, i\rangle\, y_i^{\dot{b}a} = \langle i-1|_{\dot{a}} |i\rangle^{\dot{a}}\, y_i^{\dot{b}a} = \epsilon_{\dot{a}\dot{c}} |i-1\rangle^{\dot{c}} |i\rangle^{\dot{a}}\, y_i^{\dot{b}a}$

On the left:
$\left(|i\rangle^{\dot{b}}\langle i-1|_{\dot{a}} - |i-1\rangle^{\dot{b}}\langle i|_{\dot{a}}\right) y_i^{\dot{a}a} = \epsilon_{\dot{a}\dot{c}}\left(|i\rangle^{\dot{b}} |i-1\rangle^{\dot{c}} - |i-1\rangle^{\dot{b}} |i\rangle^{\dot{c}}\right) y_i^{\dot{a}a}$.

Now, if $\dot{b} = \dot{c}$, then the expression is zero. So let's consider the cases of $\dot{b}$.

If $\dot{b} = 1$, then $\dot{c} = 2$, but then this implies that $\dot{a} = 1$ in $\epsilon_{\dot{a}\dot{c}}$;
in other words: $\dot{b} = 1 \Rightarrow \dot{a} = 1$, and similarly we can show $\dot{b} = 2 \Rightarrow \dot{a} = 2$.

In the former case, we have $\epsilon_{\dot{a}\dot{c}} = \epsilon_{\dot{1}\dot{2}} = -1$, while in the latter, $\epsilon_{\dot{a}\dot{c}} = \epsilon_{\dot{2}\dot{1}} = -1$. Therefore, in both cases, we have $\epsilon_{\dot{a}\dot{c}}\left(|i\rangle^{\dot{b}} |i-1\rangle^{\dot{c}} - |i-1\rangle^{\dot{b}} |i\rangle^{\dot{c}}\right) = \left(|i\rangle^{2} |i-1\rangle^{1} - |i-1\rangle^{2} |i\rangle^{1}\right) = \langle i-1, i\rangle$, while $\dot{a} = \dot{b}$. Thus the statement is proved.

Exercise

Finally, we saw that using the formula for y above, we could prove the formula for going directly from momentum twistors back to momentum space:

$$[i| = \frac{\langle i+1, i\rangle[\mu_{i-1}| + \langle i, i-1\rangle[\mu_{i+1}| + \langle i-1, i+1\rangle[\mu_i|}{\langle i-1, i\rangle\langle i, i+1\rangle}.$$

Exercise: Suppose we want to write this as a matrix transformation from $[\mu|$ to $[i|$. Write the matrix for the case of $n = 4$.

Exercise

Finally, we saw that using the formula for y above, we could prove the formula for going directly from momentum twistors back to momentum space:

$$[i| \;=\; \frac{\langle i+1, i\rangle[\mu_{i-1}| + \langle i, i-1\rangle[\mu_{i+1}| + \langle i-1, i+1\rangle[\mu_i|}{\langle i-1, i\rangle\langle i, i+1\rangle}\,.$$

Exercise: Suppose we want to write this as a matrix transformation from $[\mu|$ to $[i|$. Write the matrix for the case of $n = 4$.

Hint: $[2| \;=\; \frac{\langle 3,2\rangle[\mu_1|+\langle 2,1\rangle[\mu_3|+\langle 1,3\rangle[\mu_2|}{\langle 1,2\rangle\langle 2,3\rangle}$, and shift everything by one for each row.

Exercise

Finally, we saw that using the formula for y above, we could prove the formula for going directly from momentum twistors back to momentum space:

$$[i| = \frac{\langle i+1, i\rangle[\mu_{i-1}| + \langle i, i-1\rangle[\mu_{i+1}| + \langle i-1, i+1\rangle[\mu_i|}{\langle i-1, i\rangle\langle i, i+1\rangle}.$$

Exercise: Suppose we want to write this as a matrix transformation from $[\mu|$ to $[i|$. Write the matrix for the case of $n = 4$.

Hint: $[2| = \frac{\langle 3,2\rangle[\mu_1|+\langle 2,1\rangle[\mu_3|+\langle 1,3\rangle[\mu_2|}{\langle 1,2\rangle\langle 2,3\rangle}$, and shift everything by one for each row.

Answer:

$$\begin{pmatrix}[1\\ [2\\ [3\\ [4\end{pmatrix} = \begin{pmatrix} \frac{\langle 4,2\rangle}{\langle 4,1\rangle\langle 1,2\rangle} & -\frac{1}{\langle 1,2\rangle} & 0 & -\frac{1}{\langle 4,1\rangle} \\ -\frac{1}{\langle 1,2\rangle} & \frac{\langle 1,3\rangle}{\langle 1,2\rangle\langle 2,3\rangle} & -\frac{1}{\langle 2,3\rangle} & 0 \\ 0 & -\frac{1}{\langle 2,3\rangle} & \frac{\langle 2,4\rangle}{\langle 2,3\rangle\langle 3,4\rangle} & -\frac{1}{\langle 3,4\rangle} \\ -\frac{1}{\langle 4,1\rangle} & 0 & -\frac{1}{\langle 3,4\rangle} & \frac{\langle 3,1\rangle}{\langle 3,4\rangle\langle 4,1\rangle} \end{pmatrix} \begin{pmatrix}[\mu_1\\ [\mu_2\\ [\mu_3\\ [\mu_4\end{pmatrix}$$

Recap: Goal of Paper

The goal of the paper is to relate Grassmannian integral formulas written in 3 different formulations:

Twistor Space:

$$\tilde{\mathcal{L}}_{n;k}(\mathcal{W}) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{4\tilde{k}|4\tilde{k}}(B\cdot\mathcal{W})}{m_1 m_2 \ldots m_n}\,.$$

Momentum Space:

$$\mathcal{L}_{n;k}(\lambda,\tilde{\lambda},\tilde{\eta}) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{2\tilde{k}}(B_{\alpha i}\,\tilde{\lambda}_i)\,\delta^{2(n-\tilde{k})}(B^{\perp}_{\alpha i}\lambda_i)\,\delta^{(4\tilde{k})}(B_{\alpha i}\,\tilde{\eta}_{iA})}{m_1 m_2\cdots m_n}\,,$$

Momentum Twistor Space:

$$\mathcal{L}_{n;k}(\mathcal{Z}) = \mathcal{A}_n^{\mathrm{MHV}} \int \frac{d^{k\times n}C}{GL(k)} \frac{\delta^{4k|4k}(C\cdot\mathcal{Z})}{M_1 M_2\cdots M_n}\,.$$

For illustrative purposes, we do an example of how to actually compute something with the momentum twistor space Grassmannian integral formula:

$$\mathcal{L}_{n;k}(\mathcal{Z}) = \mathcal{A}_n^{\text{MHV}} \int \frac{d^{k\times n}C}{GL(k)} \frac{\delta^{4k|4k}(C\cdot\mathcal{Z})}{M_1 M_2 \cdots M_n}\,.$$

We do the simplest nontrivial case, which is NMHV. This means $k=1$, so the formula is a contour integral in $Gr(1,n)$. Written explicitly, the elements are:

$$GL(1)\backslash C = GL(1)\backslash \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix},$$

where c_i are complex numbers.

The Grassmannian integral is $\mathcal{L}_{n;1}(\mathcal{Z}) = \mathcal{A}_n^{\text{MHV}} \mathcal{I}_{n;1}(\mathcal{Z})$ with

$$\mathcal{I}_{n;1}^{(\Gamma)}(\mathcal{Z}) := \oint_\Gamma \frac{d^{1\times n}C}{GL(1)\, c_1 c_2 \cdots c_n}\, \delta^4(c_i Z_i)\, \delta^{(4)}(c_i \eta_i)\,.$$

The oriented volume form on $\mathbb{C}^n$ is $d^n C = \bigwedge_{i=1}^{n} dc^i$.

Since the bosonic delta function $\delta^4(c_i Z_i)$ fixes four c_i's, and the $GL(1)$ redundancy fixes another, the integral is over $n-5$ variables.

We suppose the contour Γ encircles a pole where exactly $n-5$ of the c_i's vanish. Such a contour can be characterized by specifying which five c_i's are non-vanishing at the pole. Let us denote these five non-vanishing c_i's by c_a, c_b, c_c, c_d, and c_e, and the corresponding contour γ_{abcde}. For example, if $n = 7$, we might have something like

$$\begin{bmatrix} c_a & 0 & c_b & c_c & c_d & c_e & 0 \end{bmatrix},$$

Thus, if we take the residue where all c_i vanish for $i \neq a, b, c, d, e$, this is, up to a sign (note: the only place where a c_i that we take the residue of shows up is in the delta functions, where they go to zero):

$$\mathcal{I}_{n;1}^{\gamma_{abcde}}(\mathcal{Z}) = \frac{\delta^4\left(c_a Z_a + c_b Z_b + c_c Z_c + c_d Z_d + c_e Z_e\right)\delta^{(4)}\left(c_a \eta_a + c_b \eta_b + c_c \eta_c + c_d \eta_d + c_e \eta_e\right)}{c_a c_b c_c c_d c_e}.$$

$$\mathcal{I}_{n;1}^{\gamma_{abcde}}(\mathcal{Z}) = \frac{\delta^4\left(c_a Z_a + c_b Z_b + c_c Z_c + c_d Z_d + c_e Z_e\right)\delta^{(4)}\left(c_a \eta_a + c_b \eta_b + c_c \eta_c + c_d \eta_d + c_e \eta_e\right)}{c_a c_b c_c c_d c_e}\,.$$

Next, we use the 5-term Schouten identity (or Cramer's rule) that states that five 4-component vectors are necessarily linearly dependent:

$$\langle ijkl\rangle\, Z_m + \langle jklm\rangle\, Z_i + \langle klmi\rangle\, Z_j + \langle lmij\rangle\, Z_k + \langle mijk\rangle\, Z_l = 0\,.$$

The 4-brackets are the fully antisymmetric $SU(2,2)$-invariants

$$\langle ijkl\rangle := -\epsilon_{\mathrm{ABCD}} Z_i^{\mathrm{A}} Z_j^{\mathrm{B}} Z_k^{\mathrm{C}} Z_l^{\mathrm{D}} = \det\left(Z_i Z_j Z_k Z_l\right).$$

Note: we can prove the 5-term Schouten using the same method as for the 3-term Schouten.
Quiz: What does this imply the c_i's are?

$$\mathcal{I}_{n;1}^{\gamma_{abcde}}(\mathcal{Z}) = \frac{\delta^4\left(c_a Z_a + c_b Z_b + c_c Z_c + c_d Z_d + c_e Z_e\right)\delta^{(4)}\left(c_a\eta_a + c_b\eta_b + c_c\eta_c + c_d\eta_d + c_e\eta_e\right)}{c_a c_b c_c c_d c_e}\,.$$

Next, we use the 5-term Schouten identity (or Cramer's rule) that states that five 4-component vectors are necessarily linearly dependent:

$$\langle ijkl\rangle\, Z_m + \langle jklm\rangle\, Z_i + \langle klmi\rangle\, Z_j + \langle lmij\rangle\, Z_k + \langle mijk\rangle\, Z_l = 0\,.$$

The 4-brackets are the fully antisymmetric $SU(2,2)$-invariants

$$\langle ijkl\rangle := -\epsilon_{\mathrm{ABCD}} Z_i^{\mathrm{A}} Z_j^{\mathrm{B}} Z_k^{\mathrm{C}} Z_l^{\mathrm{D}} = \det\left(Z_i Z_j Z_k Z_l\right).$$

Note: we can prove the 5-term Schouten using the same method as for the 3-term Schouten.
Quiz: What does this imply the c_i's are?
Answer: The constraint enforced by the bosonic delta-function is solved by

$$c_a = \langle bcde\rangle\,,\quad c_b = \langle cdea\rangle\,,\quad c_c = \langle deab\rangle\,,\quad c_d = \langle eabc\rangle\,,\quad c_e = \langle abcd\rangle\,,$$

Thus, the final answer is:

$$\mathcal{I}_{n;1}^{\gamma_{abcde}} = \frac{\delta^{(4)}\Big(\langle bcde\rangle\, \eta_a + \langle cdea\rangle\, \eta_b + \langle deab\rangle\, \eta_c + \langle eabc\rangle\, \eta_d + \langle abcd\rangle\, \eta_e \Big)}{\langle bcde\rangle\, \langle cdea\rangle\, \langle deab\rangle\, \langle eabc\rangle\, \langle abcd\rangle} =: [abcde]\,,$$

or:

$$\mathcal{L}_{n;1}^{\gamma_{abcde}} = \mathcal{A}_n^{\mathrm{MHV}}\,[abcde]\,,$$

We recognize these 5-brackets that are the building blocks of NMHV amplitudes. If we only consider 5-brackets of the form $[i, j-1, j, k-1, k] := R_{ijk}$, these are the R-invariants that you get from calculating the NMHV amplitude using super-BCFW.

For example, $\mathcal{A}_6^{\mathrm{NMHV}} = \mathcal{A}_6^{\mathrm{MHV}}(R_{135} + R_{136} + R_{146})$.

In general, the formula is:

$$\mathcal{A}_n^{\mathrm{NMHV}} = \mathcal{A}_n^{\mathrm{MHV}} \sum_{j=2}^{n-3} \sum_{k=j+2}^{n-1} R_{njk}$$

First, let's discuss how to go from Twistor Space:

$$\tilde{\mathcal{L}}_{n;k}(\mathcal{W}) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})}\, \frac{\delta^{4\tilde{k}|4\tilde{k}}(B\cdot\mathcal{W})}{m_1 m_2 \ldots m_n}\,.$$

to Momentum Space:

$$\mathcal{L}_{n;k}\big(\lambda,\tilde{\lambda},\tilde{\eta}\big) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})}\, \frac{\delta^{2\tilde{k}}\big(B_{\alpha i}\,\tilde{\lambda}_i\big)\,\delta^{2(n-\tilde{k})}\big(B^{\perp}_{\alpha i}\lambda_i\big)\,\delta^{(4\tilde{k})}\big(B_{\alpha i}\,\tilde{\eta}_{iA}\big)}{m_1 m_2\cdots m_n}\,,$$

Recall that twistor space is obtained from momentum space via a Fourier transform of λ_j, formally via

$$\int d^2\lambda_j\, \exp\big(-i\tilde{\mu}_j^a\lambda_{ja}\big)\,\bullet\,,$$

for each $j = 1, 2, \ldots, n$.

Thus, all we need to do is to do the inverse Fourier transform on all the $\tilde{\mu}_i$'s:

$$\mathcal{L}_{n;k}\big(\lambda,\tilde{\lambda},\tilde{\eta}\big) = \left(\prod_{i=1}^{n}\int d^2\tilde{\mu}_i\; e^{i\lambda_i.\tilde{\mu}_i}\right)\widetilde{\mathcal{L}}_{n;k}\big(\mathcal{W}\big)\,.$$

Let's write out $\tilde{\mathcal{L}}_{n;k}(\mathcal{W})$ a bit more explicitly:

$$\begin{aligned}
\tilde{\mathcal{L}}_{n;k}(\mathcal{W}) &= \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{4\tilde{k}|4\tilde{k}}\left(B\cdot\mathcal{W}\right)}{m_1 m_2 \ldots m_n} \\
&= \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\prod_{\alpha=1}^{\tilde{k}} \delta^4(\sum_i B_{\alpha i} W_i)\delta^{(4)}(\sum_i B_{\alpha i}\tilde{\eta}_i)}{m_1 m_2 \ldots m_n} \\
&= \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{2\tilde{k}}(\sum_i B_{\alpha i}\tilde{\lambda}_i)\delta^{2\tilde{k}}(\sum_i B_{\alpha i}\tilde{\mu}_i)\delta^{(4\tilde{k})}(\sum_i B_{\alpha i}\tilde{\eta}_i)}{m_1 m_2 \ldots m_n}
\end{aligned}$$

and let's compare side-by-side with:

$$\mathcal{L}_{n;k}\left(\lambda,\tilde{\lambda},\tilde{\eta}\right) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{2\tilde{k}}\left(B_{\alpha i}\,\tilde{\lambda}_i\right)\,\delta^{2(n-\tilde{k})}\left(B^{\perp}_{\alpha i}\lambda_i\right)\,\delta^{(4\tilde{k})}\left(B_{\alpha i}\,\tilde{\eta}_{iA}\right)}{m_1 m_2 \cdots m_n}\,,$$

Rewriting, we want to calculate

$$\left(\prod_{i=1}^{n}\int d^2\tilde{\mu}_i\; e^{i\lambda_i.\tilde{\mu}_i}\right)\int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{2\tilde{k}}(\sum_i B_{\alpha i}\tilde{\lambda}_i)\delta^{2\tilde{k}}(\sum_i B_{\alpha i}\tilde{\mu}_i)\delta^{(4\tilde{k})}(\sum_i B_{\alpha i}\tilde{\eta}_i)}{m_1 m_2 \ldots m_n}$$

so we can focus on the $\tilde{\mu}$-dependent red part:

Exercise: Do the inverse Fourier transform of the part in red.

The trick is to rewrite $\delta^{2\tilde{k}}(\sum_i B_{\alpha i}\tilde{\mu}_i)$ using the fact that this says $\tilde{\mu} \perp B$, but this is equivalent to saying: $\tilde{\mu} \subseteq B^{\perp}$.

Quiz: How do we find $B^{\perp}$ from B? Suppose we have use the GL(2) redundancy to set $B \in Gr(2,5)$ to be of the form:

$$B = \begin{pmatrix} 1 & 0 & b_{13} & b_{14} & b_{15} \\ 0 & 1 & b_{23} & b_{24} & b_{25} \end{pmatrix}$$

What's $B^{\perp}$?

The trick is to rewrite $\delta^{2\tilde{k}}(\sum_i B_{\alpha i}\tilde{\mu}_i)$ using the fact that this says $\tilde{\mu} \perp B$, but this is equivalent to saying: $\tilde{\mu} \subseteq B^{\perp}$.
Quiz: How do we find $B^{\perp}$ from B? Suppose we have use the GL(2) redundancy to set $B \in Gr(2,5)$ to be of the form:

$$B = \begin{pmatrix} 1 & 0 & b_{13} & b_{14} & b_{15} \\ 0 & 1 & b_{23} & b_{24} & b_{25} \end{pmatrix}$$

What's $B^{\perp}$?
Answer:

$$B^{\perp} = \begin{pmatrix} -B^T & Id \end{pmatrix} = \begin{pmatrix} -b_{13} & -b_{23} & 1 & 0 & 0 \\ -b_{14} & -b_{24} & 0 & 1 & 0 \\ -b_{15} & -b_{25} & 0 & 0 & 1 \end{pmatrix}$$

Exercise: In order to solve the previous exercise, use

$$\delta^{2\tilde{k}}\left(B_{\alpha i}\tilde{\mu}_i\right) = \int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\, \delta^{2n}\left(\tilde{\mu}_i - \sigma_{\bar{\alpha}} B^{\perp}_{\bar{\alpha} i}\right),$$

Show that this formula is true in the case of B above. Doing this will be helpful for understanding the dimensions of the objects we are dealing with.

Answer to 2nd exercise

The delta function is $\delta^{2\tilde{k}}(B_{\alpha i}\tilde{\mu}_i)$.

The 2 comes from the fact that each $\tilde{\mu}_i$ has 2 components, which we call x and y: $\tilde{\mu}_i = \begin{pmatrix} \tilde{\mu}_i^x \\ \tilde{\mu}_i^y \end{pmatrix}$. With our choice of gauge above, we have:

$$\delta^{2\tilde{k}}(B_{\alpha i}\tilde{\mu}_i) \Leftrightarrow \begin{cases} \tilde{\mu}_1 + b_{13}\tilde{\mu}_3 + b_{14}\tilde{\mu}_4 + b_{15}\tilde{\mu}_5 \\ \tilde{\mu}_2 + b_{23}\tilde{\mu}_3 + b_{24}\tilde{\mu}_4 + b_{25}\tilde{\mu}_5 \end{cases}$$

This is of course 4 equations because we have $\tilde{\mu}_i^x$ and $\tilde{\mu}_i^y$ as noted above.

Let's show $\delta^{2\tilde{k}}\left(B_{\alpha i}\tilde{\mu}_i\right) = \int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\, \delta^{2n}\left(\tilde{\mu}_i - \sigma_{\bar{\alpha}} B^{\perp}_{\bar{\alpha} i}\right)$, by evaluating against $\int d^{2\tilde{k}}\tilde{\mu}\, f(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5)$:

$$\int d\tilde{\mu}_1 d\tilde{\mu}_2\, \delta^{2\tilde{k}}\left(B_{\alpha i}\tilde{\mu}_i\right) f(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5)$$

Answer to 2nd exercise

The delta function is $\delta^{2\tilde{k}}(B_{\alpha i}\tilde{\mu}_i)$.

The 2 comes from the fact that each $\tilde{\mu}_i$ has 2 components, which we call x and y: $\tilde{\mu}_i = \begin{pmatrix} \tilde{\mu}_i^x \\ \tilde{\mu}_i^y \end{pmatrix}$. With our choice of gauge above, we have:

$\delta^{2\tilde{k}}(B_{\alpha i}\tilde{\mu}_i) \Leftrightarrow \begin{cases} \tilde{\mu}_1 + b_{13}\tilde{\mu}_3 + b_{14}\tilde{\mu}_4 + b_{15}\tilde{\mu}_5 \\ \tilde{\mu}_2 + b_{23}\tilde{\mu}_3 + b_{24}\tilde{\mu}_4 + b_{25}\tilde{\mu}_5 \end{cases}$

This is of course 4 equations because we have $\tilde{\mu}_i^x$ and $\tilde{\mu}_i^y$ as noted above.

Let's show $\delta^{2\tilde{k}}(B_{\alpha i}\tilde{\mu}_i) = \int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\ \delta^{2n}(\tilde{\mu}_i - \sigma_{\bar{\alpha}} B^{\perp}_{\bar{\alpha} i})$, by evaluating against $\int d^{2\tilde{k}}\tilde{\mu}\ f(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5)$:

$\int d\tilde{\mu}_1 d\tilde{\mu}_2\ \delta^{2\tilde{k}}(B_{\alpha i}\tilde{\mu}_i) f(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5)$
$= f(-b_{13}\tilde{\mu}_3 - b_{14}\tilde{\mu}_4 - b_{15}\tilde{\mu}_5, -b_{23}\tilde{\mu}_3 - b_{24}\tilde{\mu}_4 - b_{25}\tilde{\mu}_5,\ \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5)$, where we simply substituted using the equations above.

Now let's evaluate the other side:
$\int d\tilde{\mu}_1 d\tilde{\mu}_2\ \int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\ \delta^{2n}(\tilde{\mu}_i - \sigma_{\bar{\alpha}} B^{\perp}_{\bar{\alpha} i}) f(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5)$

Note that: $\delta^{2n}(\tilde{\mu}_i - \sigma_{\bar{\alpha}} B^{\perp}_{\bar{\alpha} i}) =$

$$\delta\Big(\tilde{\mu}_1 - \sigma_1(-b_{13}) - \sigma_2(-b_{14}) - \sigma_3(-b_{15})\Big)\delta\Big(\tilde{\mu}_2 - \sigma_1(-b_{23}) - \sigma_2(-b_{24}) - \sigma_3(-b_{25})\Big)\delta(\tilde{\mu}_3 - \sigma_1)\delta(\tilde{\mu}_4 - \sigma_2)\delta(\tilde{\mu}_5 - \sigma_3)$$

Then:
$\int d\tilde{\mu}_1 d\tilde{\mu}_2\ \int d^6\sigma_{\bar{\alpha}}\ \delta\Big(\tilde{\mu}_1 + b_{13}\sigma_1 + b_{14}\sigma_2 + b_{15}\sigma_3\Big)\delta\Big(\tilde{\mu}_2 + b_{23}\sigma_1 + b_{24}\sigma_2 + b_{25}\sigma_3\Big)\delta(\tilde{\mu}_3 - \sigma_1)\delta(\tilde{\mu}_4 - \sigma_2)\delta(\tilde{\mu}_5 - \sigma_3) f(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5)$

$= \int d^6\sigma_{\bar{\alpha}}\ \delta(\tilde{\mu}_3 - \sigma_1)\delta(\tilde{\mu}_4 - \sigma_2)\delta(\tilde{\mu}_5 - \sigma_3) f\big(-b_{13}\sigma_1 - b_{14}\sigma_2 - b_{15}\sigma_3, -b_{23}\sigma_1 - b_{24}\sigma_2 - b_{25}\sigma_3, \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5\big)$

$= f(-b_{13}\tilde{\mu}_3 - b_{14}\tilde{\mu}_4 - b_{15}\tilde{\mu}_5, -b_{23}\tilde{\mu}_3 - b_{24}\tilde{\mu}_4 - b_{25}\tilde{\mu}_5,\ \tilde{\mu}_3, \tilde{\mu}_4, \tilde{\mu}_5)$

Answer to 1st exercise

Coming back to the calculation of:

$$\left(\prod_{i=1}^{n}\int d^2\tilde{\mu}_i\ e^{i\lambda_i\cdot\tilde{\mu}_i}\right)\delta^{2\tilde{k}}(\sum_i B_{\alpha i}\tilde{\mu}_i)$$

Using our previous exercise/trick to rewrite this as:

$$\left(\prod_{i=1}^{n}\int d^2\tilde{\mu}_i\ e^{i\lambda_i\cdot\tilde{\mu}_i}\right)\int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\ \delta^{2n}(\tilde{\mu}_i-\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i})\ ,$$

and furthermore rewriting

$$\delta^{2n}(\tilde{\mu}_i-\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i})=\prod_{i=1}^{n}\int d^2\tau_i e^{-i\tau_i(\tilde{\mu}_i-\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i})}$$

we get:

Answer to 1st exercise

Coming back to the calculation of:

$$\left(\prod_{i=1}^{n}\int d^2\tilde{\mu}_i\; e^{i\lambda_i\cdot\tilde{\mu}_i}\right)\delta^{2\tilde{k}}(\sum_i B_{\alpha i}\tilde{\mu}_i)$$

Using our previous exercise/trick to rewrite this as:

$$\left(\prod_{i=1}^{n}\int d^2\tilde{\mu}_i\; e^{i\lambda_i\cdot\tilde{\mu}_i}\right)\int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\;\delta^{2n}(\tilde{\mu}_i-\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i})\;,$$

and furthermore rewriting

$$\delta^{2n}(\tilde{\mu}_i-\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i})=\prod_{i=1}^{n}\int d^2\tau_i e^{-i\tau_i(\tilde{\mu}_i-\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i})}$$

we get:

$$\prod_{i=1}^{n}\int d^2\tau_i\int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\left(\int d^2\tilde{\mu}_i\, e^{i(\lambda_i-\tau_i)\tilde{\mu}_i}\right)e^{i\tau_i\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i}}$$

$$=\prod_{i=1}^{n}\int d^2\tau_i\int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\left(\delta^{2n}(\lambda_i-\tau_i)\right)e^{i\tau_i\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i}}$$

$$=\prod_{i=1}^{n}\int d^{2(n-\tilde{k})}\sigma_{\bar{\alpha}}\; e^{i\lambda_i\sigma_{\bar{\alpha}}B^{\perp}_{\bar{\alpha}i}}$$

$$=\prod_{i=1}^{n}\delta^{2(n-\tilde{k})}(\lambda_i B^{\perp}_{\bar{\alpha}i})$$

which is exactly what we needed to show

Now that we have obtained the Momentum Space version:

$$\mathcal{L}_{n;k}\big(\lambda, \tilde{\lambda}, \tilde{\eta}\big) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})}\, \frac{\delta^{2\tilde{k}}\big(B_{\alpha i}\,\tilde{\lambda}_i\big)\,\delta^{2(n-\tilde{k})}\big(B^{\perp}_{\alpha i}\lambda_i\big)\,\delta^{(4\tilde{k})}\big(B_{\alpha i}\,\tilde{\eta}_{iA}\big)}{m_1 m_2 \cdots m_n}\,,$$

our final step is to get to the Momentum Twistor Space version:

$$\mathcal{L}_{n;k}(\mathcal{Z}) = \mathcal{A}_n^{\mathrm{MHV}} \int \frac{d^{k\times n}C}{GL(k)}\, \frac{\delta^{4k|4k}\big(C\cdot\mathcal{Z}\big)}{M_1 M_2 \cdots M_n}$$

$$= \mathcal{A}_n^{\mathrm{MHV}} \int \frac{d^{k\times n}C_{\hat{\alpha}i}}{GL(k)}\, \frac{\delta^{2k}\big(C_{\hat{\alpha}i}\lambda_i\big)\,\delta^{2k}\big(C_{\hat{\alpha}i}\mu_i\big)\,\delta^{(4k)}\big(C_{\hat{\alpha}i}\eta_i\big)}{M_1 M_2 \cdots M_n}$$

Overview

This is a somewhat involved calculation. Here are the main steps:

-reduce the integral from Gr(k+2,n) to Gr(k,n)

-change to momentum twistor space

-fix translational redundancy

-change variables from B to C

In order to reduce the Grassmannian by dimension 2, we start by taking the bosonic delta functions $\delta^{2(n-\tilde{k})}\left(B^{\perp} \cdot \lambda\right)$; these require that the λ 2-plane lies in the orthogonal complement of $B^{\perp}$, so using the same trick as above, we rewrite it as:

$$\prod_{\beta=1}^{\tilde{k}} \int d^2\rho_\beta \, \delta^{2n}\left(\lambda_j - \rho_\alpha B_{\alpha j}\right),$$

where ρ_α^a is a $2 \times \tilde{k}$ array of dummy integration variables.
We can use 2 dimensions of the $GL(k+2)$ to fix 2 dimensions/rows of the B-matrix, or equivalently (I don't have a proof) we can fix the $2 \times \tilde{k}$ array of ρ variables.
Exercise: What happens to the B-matrix if you set $\rho = \left(\begin{array}{ccccc} 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{array}\right)$?
Do an example with 3 columns ($\tilde{k} = 3$), so $\rho = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $n = 3$. Given this change to the B-matrix, rewrite the $\mathcal{L}_{n;k}\left(\lambda, \tilde{\lambda}, \tilde{\eta}\right)$ integral in the new form.

Answer: The $2n$ delta functions then fix the last two rows of B to be the λ's:

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \cdots & B_{kn} \\ \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_n^1 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \end{pmatrix}.$$

For the three-column example (so $\tilde{k} = 3$) with $n = 3$, $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

Then $\delta^{2n}(\lambda_j - \rho_\alpha B_{\alpha j}) =$
$\delta^2(\lambda_1 - \rho_1 b_{11} - \rho_2 b_{21} - \rho_3 b_{31})\delta^2(\lambda_2 - \rho_1 b_{12} - \rho_2 b_{22} - \rho_3 b_{32})\delta^2(\lambda_3 - \rho_1 b_{13} - \rho_2 b_{23} - \rho_3 b_{33})$

where the 2's are because there are both x and y components. Writing them explicitly, while also substituting in the values for ρ we fixed above (either 0 or 1), we get:

$$\delta(\lambda_1^x - b_{21})\delta(\lambda_1^y - b_{31})\delta(\lambda_2^x - b_{22})\delta(\lambda_2^y - b_{32})\delta(\lambda_3^x - b_{23})\delta(\lambda_3^y - b_{33})$$

This just sets $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \mapsto \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \lambda_1^x & \lambda_2^x & \lambda_3^x \\ \lambda_1^y & \lambda_2^y & \lambda_3^y \end{pmatrix}$

This determines the $\delta^{2(n-\tilde{k})}\left(B^{\perp} \cdot \lambda\right)$ delta functions. To see what happens to the other delta functions, we realize that only the last 2 rows of B change, so we can separate out these delta functions.

For these, we find that the last two rows of $\delta^{2\tilde{k}}\left(B_{\alpha i}\, \tilde{\lambda}_i\right) \delta^{(4\tilde{k})}\left(B_{\alpha i}\, \tilde{\eta}_{iA}\right)$ simply give momentum conservation and supermomentum conservation.

Thus, after evaluating the ρ integrals, we find

$$\mathcal{L}_{n;k}\left(\lambda, \tilde{\lambda}, \tilde{\eta}\right) = \delta^4\left(\lambda_i \tilde{\lambda}_i\right) \delta^{(8)}\left(\lambda_i \tilde{\eta}_i\right) \times \int \frac{d^{k\times n} B_{\hat{\alpha} i}}{GL(k) \ltimes T_k} \frac{\delta^{2k}\left(B_{\hat{\alpha} i} \tilde{\lambda}_i\right) \delta^{(4k)}\left(B_{\hat{\alpha} i} \tilde{\eta}_i\right)}{m_1 m_2 \cdots m_n},$$

with $\hat{\alpha} = 1, 2, \ldots, k$.

T_k indicates the translational redundancy in the $B_{\hat{\alpha}i}$-variables. The translational symmetry acts as

$$B_{\hat{\alpha}i} \to B_{\hat{\alpha}i} + r_{x\hat{\alpha}}\lambda_i^x + r_{y\hat{\alpha}}\lambda_i^y \qquad \text{for all } i \text{ simultaneously,}$$

where $r_{1\hat{\alpha}}$ and $r_{2\hat{\alpha}}$ are any numbers.

Exercise: Show that this leaves the integral unchanged.

Answer: This is a mixing of the last two rows in the B-matrix with the other rows, and this leaves the minors m_i unchanged. It is also clear that on the support of the two delta functions $\delta^4(\lambda_i\tilde{\lambda}_i)\delta^{(8)}(\lambda_i\tilde{\eta}_i)$ (that encode momentum and supermomentum conservation), the delta-functions in the integral are invariant under such a shift.

In detail, if we look at: $\delta^{2k}(B_{\hat{\alpha}i}\tilde{\lambda}_i)\,\delta^{(4k)}(B_{\hat{\alpha}i}\tilde{\eta}_i)$, or just one of the terms of the first one (the others are analogous), we find that

$B_{\hat{\alpha}1}\tilde{\lambda}_1 + B_{\hat{\alpha}2}\tilde{\lambda}_2 + \cdots + B_{\hat{\alpha}n}\tilde{\lambda}_n = 0$
$\mapsto$
$(B_{\hat{\alpha}1} + r_{x\hat{\alpha}}\lambda_1^x + r_{y\hat{\alpha}}\lambda_1^y)\tilde{\lambda}_1 + (B_{\hat{\alpha}2} + r_{x\hat{\alpha}}\lambda_2^x + r_{y\hat{\alpha}}\lambda_2^y)\tilde{\lambda}_2 + \cdots + (B_{\hat{\alpha}n} + r_{x\hat{\alpha}}\lambda_n^x + r_{y\hat{\alpha}}\lambda_n^y)\tilde{\lambda}_n$
$= B_{\hat{\alpha}1}\tilde{\lambda}_1 + B_{\hat{\alpha}2}\tilde{\lambda}_2 + \cdots + B_{\hat{\alpha}n}\tilde{\lambda}_n + r_{x\hat{\alpha}}(\lambda_1^x\tilde{\lambda}_1 + \lambda_2^x\tilde{\lambda}_2 + \cdots \lambda_n^x\tilde{\lambda}_n) + r_{y\hat{\alpha}}(\lambda_1^y\tilde{\lambda}_1 + \lambda_2^y\tilde{\lambda}_2 + \cdots \lambda_n^y\tilde{\lambda}_n)$,
and the latter two terms are zero since $\lambda^a\tilde{\lambda}^b = 0$ for $a, b = 1, 2$ by momentum conservation.

We change variables in the external data to go from momentum space to momentum twistor space. The momentum supertwistors $\mathcal{Z}_i = (\lambda_i, \mu_i \,|\, \eta_i)$ are related to the momentum space variables via the relations we gave at the beginning for the formula of $[i|$. Using these relations, we directly find for each $\hat{\alpha} = 1, 2, \ldots, k$:

$$\sum_{i=1}^{n} B_{\hat{\alpha}i}\tilde{\lambda}_i = -\sum_{i=1}^{n} C_{\hat{\alpha}i}\mu_i \,, \qquad \sum_{i=1}^{n} B_{\hat{\alpha}i}\tilde{\eta}_i = -\sum_{i=1}^{n} C_{\hat{\alpha}i}\eta_i \,,$$

Exercise: find what the $C_{\hat{\alpha}i}$ are equal to.

Answer:
$C_{\hat{\alpha}i}$ is the coefficient of μ_i. If, in $\sum_{i=1}^n B_{\hat{\alpha}i}\tilde{\lambda}_i$ we look at $B_{\hat{\alpha}i}[i|$, this is equal to, by the formula for $[i|$:

$B_{\hat{\alpha}i}\left(\frac{\langle i,i+1\rangle[\mu_{i-1}|+\langle i-1,i\rangle[\mu_{i+1}|+\langle i+1,i-1\rangle[\mu_i|}{\langle i-1,i\rangle\langle i,i+1\rangle}\right)$, and the piece with $[\mu_i|$ is $\frac{B_{\hat{\alpha},i}\langle i+1,i-1\rangle}{\langle i-1,i\rangle\langle i,i+1\rangle}$.

The only other terms with a $[\mu_i|$ are from $[i-1|$ and $[i+1|$, so we if look at these three terms summed together:

$$\left(\frac{B_{\hat{\alpha},i-1}\langle i-2,i-1\rangle}{\langle i-2,i-1\rangle\langle i-1,i\rangle}+\frac{B_{\hat{\alpha},i}\langle i+1,i-1\rangle}{\langle i-1,i\rangle\langle i,i+1\rangle}+\frac{B_{\hat{\alpha},i+1}\langle i+1,i+2\rangle}{\langle i,i+1\rangle\langle i+1,i+2\rangle}\right)[\mu_i|$$

Thus,

$$C_{\hat{\alpha}i}=\frac{\langle i,i+1\rangle B_{\hat{\alpha},i-1}+\langle i-1,i\rangle B_{\hat{\alpha},i+1}+\langle i+1,i-1\rangle B_{\hat{\alpha}i}}{\langle i-1,i\rangle\langle i,i+1\rangle}.$$

Using a trick of Arkani-Hamed, Cachazo, and Cheung, we can rewrite the $(k+2) \times (k+2)$ minors m_i of the B-matrix in terms of the $k \times k$ minors of the C-matrix as

$$m_1 = (B_1 \ldots B_{k+2}) = -\langle 12 \, \cdots \langle k+1, k+2 \, (C_2 \ldots C_{k+1}) \quad \text{etc.}$$

Defining the $k \times k$ minors of the $k \times n$ C-matrix to be $M_1 := (C_1 \ldots C_k)$ etc, we thus have

$$m_1 m_2 \cdots m_n = (-1)^n \big(\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle \big)^{k+1} M_1 M_2 \cdots M_n \,.$$

(We drop the signs $(-1)^n$ just as we drop 2π's in the Fourier transforms.) Thus, we now have

$$\mathcal{L}_{n;k}(\lambda, \tilde{\lambda}, \tilde{\eta}) = \frac{\delta^4(\lambda_i \tilde{\lambda}_i)\, \delta^{(8)}(\lambda_i \tilde{\eta}_i)}{\big(\langle 12 \rangle \langle 23 \rangle \cdots \langle 1n \rangle \big)^{k+1}} \int \frac{d^{k \times n} B_{\hat{\alpha} i}}{GL(k) \ltimes T_k} \, \frac{\delta^{2k}(C_{\hat{\alpha} i} \mu_i)\, \delta^{(4k)}(C_{\hat{\alpha} i} \eta_i)}{M_1 M_2 \cdots M_n} .$$

It is here understood that the C's are functions of the B's as given above.

Note that the Schouten identity guarantees the following two important properties:

- The C's are invariant under the translations
  $B_{\hat{\alpha}i} \to B_{\hat{\alpha}i} + r_{1\hat{\alpha}}\lambda_i^1 + r_{2\hat{\alpha}}\lambda_i^2 \qquad \text{for all } i \text{ simultaneously},$
- The expression for the C's in terms of the B's implies that $C_{\hat{\alpha}i}\lambda_i = 0$.

We would now like to do two things: fix the translational redundancy and rewrite the integral in terms of C's instead of B's.

Because of the translational invariance, the B's are not independent variables: for example we can use translations to set $2k$ of them to zero (see below). So after fixing translational invariance, we will have $kn - 2k = k(n-2)$ variables to integrate over.

Let us use the translation invariance T_k to fix the first two columns in $B_{\hat{\alpha} i}$ to be zero.

Exercise: How do we do this, and what is the result?

Let us use the translation invariance T_k to fix the first two columns in $B_{\hat{\alpha}i}$ to be zero.
Exercise: How do we do this, and what is the result?

Answer: This means that all $\hat{\alpha} = 1, \dots, k$ we set $B_{\hat{\alpha}1} = B_{\hat{\alpha}2} = 0$, explicitly:
$B_{\hat{\alpha}1} \to B_{\hat{\alpha}1} + r_{x\hat{\alpha}}\lambda_1^x + r_{y\hat{\alpha}}\lambda_1^y = 0$
$B_{\hat{\alpha}2} \to B_{\hat{\alpha}2} + r_{x\hat{\alpha}}\lambda_2^x + r_{y\hat{\alpha}}\lambda_2^y = 0$

If we solve this, we find that $r_{x\hat{\alpha}} = \frac{B_{\hat{\alpha}2}\lambda_1^y - B_{\hat{\alpha}1}\lambda_2^y}{\langle 12 \rangle}$, and $r_{y\hat{\alpha}} = \frac{B_{\hat{\alpha}1}\lambda_2^x - B_{\hat{\alpha}2}\lambda_1^x}{\langle 12 \rangle}$, and all $B_{\hat{\alpha}3}, B_{\hat{\alpha}4}, \dots$ stay the same.

This gives

$$\frac{d^{k\times n}B_{\hat{\alpha}i}}{T_k} = \langle 12 \rangle^k \, d^{k\times(n-2)}B_{\hat{\alpha}i} \,,$$

where the included prefactor preserves the scaling properties of the measure. (This can be derived more carefully as a Jacobian of the gauge fixing.)

We know how $C_{\hat{\alpha}i}$ is related to $B_{\hat{\alpha}i}$ from above. We can use that relation to solve for $k(n-2)$ of the components of C in terms of the $k(n-2)$ unfixed components of B. Given our choice to set $B_{\hat{\alpha}1} = B_{\hat{\alpha}2} = 0$, we have the following system of $k(n-2)$ equations:

$$C_{\hat{\alpha}i} = \begin{cases} \dfrac{\langle i, i+1\rangle B_{\hat{\alpha},i-1} + \langle i+1, i-1\rangle B_{\hat{\alpha}i} + \langle i-1, i\rangle B_{\hat{\alpha},i+1}}{\langle i-1, i\rangle \langle i, i+1\rangle} & \text{for} \quad 3 < i < n \\[2ex] \dfrac{\langle 42\rangle B_{\hat{\alpha}3} + \langle 23\rangle B_{\hat{\alpha}4}}{\langle 23\rangle \langle 34\rangle} & \text{for} \quad i = 3 \\[2ex] \dfrac{\langle n1\rangle B_{\hat{\alpha},n-1} + \langle 1, n-1\rangle B_{\hat{\alpha}n}}{\langle n-1, n\rangle \langle n1\rangle} & \text{for} \quad i = n \end{cases} .$$

We can write this as $C_{\hat{\alpha}\hat{j}} = B_{\hat{\alpha}\hat{i}} Q_{\hat{i}\hat{j}}$, with $\hat{i}, \hat{j} = 3, \ldots, n$, for a square symmetric matrix $Q_{\hat{i}\hat{j}}$ with nonzero entries only on and adjacent to the main diagonal. In their Appendix, they show that

$$|\det Q\,| = \frac{\langle 12\rangle^2}{\langle 12\rangle \langle 23\rangle \cdots \langle n1\rangle} .$$

Therefore the measure transforms as

$$d^{k(n-2)} B_{\hat{\alpha}\hat{i}} = \frac{d^{k(n-2)} C_{\hat{\alpha}\hat{i}}}{|\det Q\,|^k} = \left(\frac{\langle 12\rangle \langle 23\rangle \cdots \langle n1\rangle}{\langle 12\rangle^2} \right)^k d^{k(n-2)} C_{\hat{\alpha}\hat{i}} .$$

We take the absolute value of the determinant since the overall sign is irrelevant.

With these formulas, the integral now takes the form

$$\mathcal{L}_{n;k}(\mathcal{Z}) = \frac{\delta^4(\lambda_i\tilde{\lambda}_i)\,\delta^{(8)}(\lambda_i\tilde{\eta}_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle 1n\rangle}\frac{1}{\langle 12\,^k}\int \frac{d^{k\times(n-2)}C_{\hat{\alpha}i}}{GL(k)}\,\frac{\delta^{2k}(C_{\hat{\alpha}i}\mu_i)\,\delta^{(4k)}(C_{\hat{\alpha}i}\eta_i)}{M_1M_2\cdots M_n}\bigg|_{C_{\hat{\alpha}1,2}=C^{(0)}_{\hat{\alpha}1,2}}.$$

We recognize the first factor as the MHV superamplitude $\mathcal{A}_n^{\mathrm{MHV}}$.
The restriction of $C_{\hat{\alpha}1}$ and $C_{\hat{\alpha}2}$ follows from the relation between the B and C; it was used above to solve for $k(n-2)$ components of C in terms of the B's, but the remaining $2k$ components of C are then fixed as

$$C_{\hat{\alpha}1} = \frac{B_{\hat{\alpha}n}}{\langle n1\rangle} = \sum_{j=3}^{n}\frac{\langle 2j\rangle}{\langle 12\rangle}C_{\hat{\alpha}j} =: C^{(0)}_{\hat{\alpha}1}\,, \qquad C_{\hat{\alpha}2} = \frac{B_{\hat{\alpha}3}}{\langle 23\rangle} = -\sum_{j=3}^{n}\frac{\langle 1j\rangle}{\langle 12\rangle}C_{\hat{\alpha}j} =: C^{(0)}_{\hat{\alpha}2}\,.$$

Thus, when evaluating the integral, $C_{\hat{\alpha}1}$ and $C_{\hat{\alpha}2}$ are functions of the other $k(n-2)$ C-components. This is a restriction of the region of integration that we can also impose via $2k$ delta functions $\delta\big(C_{\hat{\alpha}i} - C^{(0)}_{\hat{\alpha}i}\big)$ for $i = 1, 2$. Moreover, it follows from the explicit solution in the previous equation that the constraints are equivalent to $C_{\hat{\alpha}i}\lambda_i = 0$. Thus we can rewrite the delta function restriction as

$$\delta\big(C_{\hat{\alpha}1} - C^{(0)}_{\hat{\alpha}1}\big)\,\delta\big(C_{\hat{\alpha}2} - C^{(0)}_{\hat{\alpha}2}\big) = \langle 12\ \delta^2\big(C_{\hat{\alpha}i}\lambda_i\big)\,.$$

for each $\hat{\alpha} = 1, 2, \ldots, k$.

We then have

$$\mathcal{L}_{n;k}(\mathcal{Z}) = \mathcal{A}_n^{\text{MHV}} \int \frac{d^{k\times n} C_{\hat{\alpha}i}}{GL(k)} \frac{\delta^{2k}\left(C_{\hat{\alpha}i}\lambda_i\right) \delta^{2k}\left(C_{\hat{\alpha}i}\mu_i\right) \delta^{(4k)}\left(C_{\hat{\alpha}i}\eta_i\right)}{M_1 M_2 \cdots M_n} \,.$$

Although we chose to fix the first two columns of B to be zero, the answer is independent of that choice; the factors of $\langle 12\rangle$ cancel out. We can now write the result directly in terms of the momentum supertwistors $\mathcal{Z}_i = (Z_i|\eta_i) = (\lambda_i, \mu_i|\eta_i)$ as

$$\mathcal{L}_{n;k}(\mathcal{Z}) = \mathcal{A}_n^{\text{MHV}} \int \frac{d^{k\times n} C_{\hat{\alpha}i}}{GL(k)} \frac{\delta^{4k}\left(C_{\hat{\alpha}i}Z_i\right) \delta^{(4k)}\left(C_{\hat{\alpha}i}\eta_i\right)}{M_1 M_2 \cdots M_n} = \mathcal{A}_n^{\text{MHV}} \int \frac{d^{k\times n} C_{\hat{\alpha}i}}{GL(k)}$$

This completes our derivation of the $\mathcal{N}=4$ SYM Grassmannian integral in momentum twistor space from that in momentum space.