Part 1 of Grassmannian Integrals:

Grassmannian for scattering amplitudes in 4d $\mathcal{N} = 4$ SYM and 3d ABJM by Elvang, Huang, Keeler, Lam, Olson, Roland, and Speyer

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Geometry and Amplitudes Journal Club, Fall 2023

Grassmannian Integrals

We will be studying "Grassmannian integral" expressions that look like the following:

$$\mathcal{L}_{n,k}(\mathcal{W}_i) \;=\; \int \frac{d^{n\times k} C_{ai}}{GL(k)\, \prod_{j=1}^{n} M_j} \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{l=1}^{n} C_{al} \mathcal{W}_l^{A} \right).$$

This is the "link representation" first conjectured by Arkani-Hamed, Cachazo, Chueng, and Kaplan in 2009.

Before going into the details, we ask 2 questions:
(1) Why do we care?
(2) What does this have to do with the Grassmannian?

Why do we care?

Perhaps the a similar question is how that link formula was obtained.

We have the well-known BCFW recursion relations for obtaining amplitudes. In color-ordered planar $\mathcal{N} = 4$ SYM, each BCFW term for a tree amplitude turns out to be a Yangian invariant. So if we find a formula that generates all Yangian invariants, then this formula will generate all BCFW terms, giving us a way to get the amplitude. BCFW terms are also related to leading singularities, which are also Yangian invariant, so such a formula would also generate all leading singularities.

It was proven that for given n and k, $\mathcal{L}_{n,k}$ is the unique cyclically invariant integral-expression that generates all Yangian invariants.

Grassmannians

For the second question, it would be good to first define what a Grassmannian is:

Let V be an n-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$)

For any integer $0 \leq k \leq n$, $Gr_k(V)$ is the set of all n-dimensional linear subspaces (through the origin) of V. In the context of scattering amplitudes, n counts the number of external particles while k is related to particle helicities.

Examples:
(1) $Gr(0, n) = \{0\}$ the origin

(2) $Gr(n, n) = \mathbb{C}^n$ the whole space (is a point in the Grassmannian)

We have a natural duality: $Gr(k, n) \cong Gr(n-k, n)$ by considering the "orthogonal complement" $L^\perp$ of $L \in Gr(k, n)$.

(3) $Gr(1, n) \cong \mathbb{P}^{n-1}$

Writing Grassmannians

We tend to write $Gr(k,n)$ as $GL(k)\backslash$[full rank $k \times n$ matrices], and think about it as k row vectors spanning a plane up to invertible basis transformation inside the plane.

Each point in $Gr(k,n)$ given by $GL(k)\backslash[k \times n]$, since it is full rank, has a set of columns $\lambda \in \binom{[n]}{k}$ whose determinant is nonzero, so we can use $GL(k)$ to set λ to be the identity so $\left(k \times n\right) \cong \begin{pmatrix} 1 & * & 0 & & 0 \\ 0 & * & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & 0 & & 1 \end{pmatrix}$ where the $00\cdots1\cdots0$ columns are in λ.

What does this have to do with Grassmannians?

To motivate where the Grassmannian comes from, let's take another look at the formula:

$$\mathcal{L}_{n,k}(\mathcal{W}_i) = \int \frac{d^{n\times k} C_{ai}}{GL(k) \prod_{j=1}^{n} M_j} \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{l=1}^{n} C_{al}\mathcal{W}_l^{A}\right).$$

We see an integral over a Grassmannian; namely, the $\frac{d^{n\times k} C_{ai}}{GL(k)}$ part.

What does this have to do with Grassmannians?

Why do we have these variables C_{ai}?

One way to think about it, is that this linearizes momentum conservation. In spinor helicity, momentum conservation is the equation

$$\sum_{i=1}^{n} |i\rangle [i| = 0$$

, and we know that Lorentz transformations will rotate the 2 row vectors of each λ, $\tilde{\lambda}$ by SL(2). So this equation says that the 2-planes given by λ and $\tilde{\lambda}$ are orthogonal. This can be encoded with an auxiliary matrix C where $\lambda \subseteq C, \tilde{\lambda} \subseteq C^{\perp}$ via the delta equations: $\delta(C \cdot \tilde{\lambda})\delta(C^{\perp} \cdot \lambda)$.

So we have these variables C_{ai}, which can be assembled into a matrix. But to be a Grassmannian, we need to have that GL(k) action. Where does this come from?

We can easily compute that $\mathcal{L}_{n,k}(\mathcal{W}_i) = \int \frac{d^{n\times k} C_{\mathrm{a}i}}{\prod_{j=1}^{n} M_j} \prod_{\mathrm{a}=1}^{k} \delta^{4|4}\left(\sum_{l=1}^{n} C_{\mathrm{a}l} \mathcal{W}_l^{\mathrm{A}}\right)$ is invariant if we send $C_{ai} \mapsto M \cdot C_{ai}, M \in GL(k)$ Exercise.

Goal of Paper

The goal of the paper is to relate Grassmannian integral formulas written in 3 different formulations:

Twistor Space:

$$\tilde{\mathcal{L}}_{n;k}(\mathcal{W}) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{4\tilde{k}|4\tilde{k}}(B\cdot\mathcal{W})}{m_1 m_2 \ldots m_n}.$$

Momentum Space:

$$\mathcal{L}_{n;k}(\lambda,\tilde{\lambda},\tilde{\eta}) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{2\tilde{k}}(B_{\alpha i}\,\tilde{\lambda}_i)\,\delta^{2(n-\tilde{k})}(B^{\perp}_{\alpha i}\lambda_i)\,\delta^{(4\tilde{k})}(B_{\alpha i}\,\tilde{\eta}_{iA})}{m_1 m_2\cdots m_n},$$

Momentum Twistor Space:

$$\mathcal{L}_{n;k}(\mathcal{Z}) = \mathcal{A}_n^{\text{MHV}} \int \frac{d^{k\times n}C}{GL(k)} \frac{\delta^{4k|4k}(C\cdot\mathcal{Z})}{M_1 M_2\cdots M_n}.$$

Kinematic Spaces

Thus, let's review these different notions.
For n-particle N^{k}MHV amplitudes in 4d planar $\mathcal{N} = 4$ SYM there are three formulations of the external data:

- The **momentum space** formulation uses the spinor helicity representation for the external momenta, e.g. $p_{a\dot{b}} = \lambda_a \tilde{\lambda}_{\dot{b}}$. The relevant Grassmannian is $\mathrm{G}(k+2, n)$.
- The **twistor space** formulation encodes the data via a half-Fourier transform of the external momenta, i.e. a Fourier transform of λ, but not $\tilde{\lambda}$. This representation makes the superconformal symmetry $SU(2, 2|4)$ manifest. As in momentum space, the Grassmannian is $\mathrm{G}(k+2, n)$.
- The **momentum twistor** formulation is applicable in the planar limit and makes the *dual* superconformal symmetry $SU(2, 2|4)$ manifest. The relevant Grassmannian is $\mathrm{G}(k, n)$. If we remember the last equation on the previous slide, we can think about this -2 lower degree k in terms of the MHV factor that got stripped off: $\mathcal{A}_n^{\mathrm{MHV}}$.

Relating Kinematic Spaces

Here is a summary of how to move between the different versions:

twistor space		momentum space		momentum twistor space
$\mathcal{W}_i = (\tilde{\mu}_i, \tilde{\lambda}_i \,\vert\, \tilde{\eta}_i)$	$\underset{\substack{\text{Fourier transform} \\ \text{eq (2.6)}}}{\longleftrightarrow}$	$(\lambda_i, \tilde{\lambda}_i \,\vert\, \tilde{\eta}_i)$	$\underset{\substack{\text{incidence relations} \\ \text{eq (2.10)}}}{\longleftrightarrow}$	$\mathcal{Z}_i = (\lambda_i, \mu_i \,\vert\, \eta_i)$

Twistor Space to Momentum Space

For the first one, twistor space is obtained from momentum space via a Fourier transform of λ_j, formally via

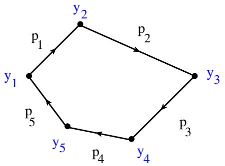
$$\int d^2\lambda_j \, \exp\left(-i\tilde{\mu}_j^a \lambda_{ja}\right) \bullet \,,$$

for each $j = 1, 2, \ldots, n$. The bullet indicates the expression that is Fourier transformed. The external data is then encoded in the 4-component twistor $W_i = (\tilde{\mu}_i, \tilde{\lambda}_i)$ and its companion, the supertwistor $\mathcal{W}_i = (\tilde{\mu}_i, \tilde{\lambda}_i \,|\, \tilde{\eta}_i)$.

Momentum Space to Momentum Twistor Space

For the second arrow, let's quickly introduce momentum twistors.

Dual space is defined by the relation:
$p_i^\mu = y_i^\mu - y_{i+1}^\mu$, or if we go to spinor-helicity notation:
$p_i^{\dot{a}a} = y_i^{\dot{a}a} - y_{i+1}^{\dot{a}a}$. We can depict this with the following picture:



Since $p_i^{\dot{a}a} = -\left|i\right\rangle^{\dot{a}}\left[i\right|^a$, if we contract both sides with $\left\langle i\right|_{\dot{a}}$, we get:
$-(\left\langle i\right|_{\dot{a}}\left|i\right\rangle^{\dot{a}}[i|^a) = 0 = \left\langle i\right|_{\dot{a}} y_i^{\dot{a}a} - \left\langle i\right|_{\dot{a}} y_{i+1}^{\dot{a}a}$

Thus, we define: $[\mu_i|^a := \left\langle i\right|_{\dot{a}} y_i^{\dot{a}a} = \left\langle i\right|_{\dot{a}} y_{i+1}^{\dot{a}a}$

The momentum twistors are denoted: $Z_i^I = (\left|i\right\rangle, [\mu_i|)$.

Exercise

The formula above shows that a point Z_i^I corresponds to a null-line in y-space (the null line passing through y_i and y_{i+1}).
Similarly, a line in Z-space determines a point in y-space. Exercise: prove this.

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Similarly, a line in Z-space determines a point in y-space. Exercise: prove this.
Hint. Prove:

$$y_i^{\dot{a}a} \;=\; \frac{|i\rangle^{\dot{a}}\,[\mu_{i-1}|^a - |i-1\rangle^{\dot{a}}\,[\mu_i|^a}{\langle i-1, i\rangle}\,.$$

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The formula above shows that a point Z_i^I corresponds to a null-line in y-space (the null line passing through y_i and y_{i+1}).
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Answer: note that the point $y_i^{\dot{a}a}$ is involved in two incidence relations: $[\mu_i| = \langle i|y_i$ and $[\mu_{i-1}| = \langle i-1|y_i$. Combining these leads to

$$|i\rangle^{\dot{b}}\,[\mu_{i-1}|^a - |i-1\rangle^{\dot{b}}\,[\mu_i|^a = \left(|i\rangle^{\dot{b}}\langle i-1|_{\dot{a}} - |i-1\rangle^{\dot{b}}\langle i|_{\dot{a}}\right) y_i^{\dot{a}a} = \langle i-1, i\rangle\, y_i^{\dot{b}a}$$

Question: typo?
$\left(|i\rangle^{\dot{b}}\langle i-1|_{\dot{a}} - |i-1\rangle^{\dot{b}}\langle i|_{\dot{a}}\right) y_i^{\dot{a}a} =$
$|i\rangle^{\dot{b}}\langle i-1|_1 y_i^{1a} - |i-1\rangle^{\dot{b}}\langle i|_1 y_i^{1a} + |i\rangle^{\dot{b}}\langle i-1|_2 y_i^{2a} - |i-1\rangle^{\dot{b}}\langle i|_2 y_i^{2a} \neq \langle i-1, i\rangle\, y_i^{\dot{b}a}$
???

Answer/Correction

$$|i\rangle^{\dot b}\,[\mu_{i-1}|^{a} - |i-1\rangle^{\dot b}\,[\mu_i|^{a} \;=\; \left(|i\rangle^{\dot b}\langle i-1|_{\dot a} - |i-1\rangle^{\dot b}\langle i|_{\dot a}\right) y_i^{\dot a a} \;=\; \langle i-1, i\rangle\, y_i^{\dot b a}$$

Question: typo???

Answer: The formula is correct. It must be, because the first comes purely from substituting the incidence relations, and we know that the solution is: $y_i^{\dot a a} = \frac{|i\rangle^{\dot a}\,[\mu_{i-1}|^{a} - |i-1\rangle^{\dot a}\,[\mu_i|^{a}}{\langle i-1, i\rangle}$, so the 2nd equality (giving the thing on the very right) must be the answer. The question is how to get there.

They key thing is to be very careful with signs and raising/lowering via the epsilon symbol (in particular, which index on the epsilon symbol matters). For this, the main formulas are:

$\epsilon^{ab} = \epsilon^{\dot a\dot b} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{ab} = -\epsilon_{\dot a\dot b}$, which gives us: $|p\rangle^{\dot a} = \epsilon^{\dot a\dot b}\langle p|_{\dot b}$ and $\langle p|_{\dot a} = \epsilon_{\dot a\dot b}\,|p\rangle^{\dot b}$.

We could do this by writing the matrices out explicitly. Because $y^{\dot a a} = y_\mu(\overline{\sigma}^{\mu})^{\dot a a}$, we would get $y^{\dot a a} = \begin{pmatrix} y_0 - y_3 & -y_1 + iy_2 \\ -y_1 - iy_2 & y_0 + y_3 \end{pmatrix}$. This would be the right side. On the left side, we would have something like $ep_{\dot a\dot c}y^{\dot a a} = -y^{a}_{\dot c}$. This turns out to be $\begin{pmatrix} -y_1 - iy_2 & y_0 + y_3 \\ -y_0 + y_3 & y_1 - iy_2 \end{pmatrix}$. We can multiply out the matrices and get the answer. But the quicker way to do this is the following.

We first rewrite everything in kets: $\langle i-1, i\rangle\, y_i^{\dot b a} = \langle i-1|_{\dot a}\,|i\rangle^{\dot a}\, y_i^{\dot b a} = \epsilon_{\dot a\dot c}\,|i-1\rangle^{\dot c}\,|i\rangle^{\dot a}\, y_i^{\dot b a}$ on the right.

On the left, $\left(|i\rangle^{\dot b}\langle i-1|_{\dot a} - |i-1\rangle^{\dot b}\langle i|_{\dot a}\right) y_i^{\dot a a} = \epsilon_{\dot a\dot c}\left(|i\rangle^{\dot b}\,|i-1\rangle^{\dot c} - |i-1\rangle^{\dot b}\,|i\rangle^{\dot c}\right) y_i^{\dot a a}$. Now, if $\dot b = \dot c$, then the expression is zero. So let's consider the cases of $\dot b$.

If $\dot b = 1$, then $\dot c = 2$, but then this implies that $\dot a = 1$ in $\epsilon_{\dot a\dot c}$; in other words: $\dot b = 1 \Rightarrow \dot a = 1$, and similarly we can show $\dot b = 2 \Rightarrow \dot a = 2$. In the former case, we have $\epsilon_{\dot a\dot c} = \epsilon_{\dot 1\dot 2} = -1$, while in the latter, $\epsilon_{\dot a\dot c} = \epsilon_{\dot 2\dot 1} = -1$. Therefore, in both cases, we have $\epsilon_{\dot a\dot c}\left(|i\rangle^{\dot b}\,|i-1\rangle^{\dot c} - |i-1\rangle^{\dot b}\,|i\rangle^{\dot c}\right) = \left(|i\rangle^{2}\,|i-1\rangle^{1} - |i-1\rangle^{2}\,|i\rangle^{1}\right) = \langle i-1, i\rangle$, while $\dot a = \dot b$. Thus the statement is proved.

Exercise

We can go directly from momentum twistors to momentum space without using dual space with the relation:

$$[i| \;=\; \frac{\langle i+1, i\rangle[\mu_{i-1}| + \langle i, i-1\rangle[\mu_{i+1}| + \langle i-1, i+1\rangle[\mu_i|}{\langle i-1, i\rangle\langle i, i+1\rangle}\,.$$

Exercise: prove.

Exercise Solution

Solution:

$$
\begin{aligned}
p_i &= y_i - y_{i+1} \\
&= \frac{|i\rangle^{\dot a}\,[\mu_{i-1}|^a - |i-1\rangle^{\dot a}\,[\mu_i|^a}{\langle i-1, i\rangle} - \frac{|i+1\rangle^{\dot a}\,[\mu_i|^a - |i\rangle^{\dot a}\,[\mu_{i+1}|^a}{\langle i, i+1\rangle} \\
&= \frac{|i\rangle^{\dot a}\,\langle i, i+1\rangle\;[\mu_{i-1}|^a - |i-1\rangle^{\dot a}\,\langle i, i+1\rangle\;[\mu_i|^a - |i+1\rangle^{\dot a}\,\langle i-1, i\rangle\;[\mu_i|^a + |i\rangle^{\dot a}\,\langle i-1, i\rangle\;[\mu_{i+1}|^a}{\langle i-1, i\rangle\langle i, i+1\rangle} \\
&= \frac{|i\rangle^{\dot a}\,\langle i, i+1\rangle\;[\mu_{i-1}|^a - \left(|i-1\rangle^{\dot a}\,\langle i, i+1\rangle\; + |i+1\rangle^{\dot a}\,\langle i-1, i\rangle\,\right)[\mu_i|^a + |i\rangle^{\dot a}\,\langle i-1, i\rangle\;[\mu_{i+1}|^a}{\langle i-1, i\rangle\langle i, i+1\rangle} \\
&= \frac{|i\rangle^{\dot a}\,\langle i, i+1\rangle\;[\mu_{i-1}|^a - \left(|i\rangle^{\dot a}\,\langle i+1, i-1\rangle\,\right)[\mu_i|^a + |i\rangle^{\dot a}\,\langle i-1, i\rangle\;[\mu_{i+1}|^a}{\langle i-1, i\rangle\langle i, i+1\rangle} \\
&= \frac{|i\rangle^{\dot a}\Big(\,\langle i, i+1\rangle\;[\mu_{i-1}|^a - \langle i+1, i-1\rangle\;[\mu_i|^a + \langle i-1, i\rangle\;[\mu_{i+1}|^a\Big)}{\langle i-1, i\rangle\langle i, i+1\rangle}
\end{aligned}
$$

where we used the Schouten identity $|i-1\rangle\,\langle i, i+1\rangle\; + |i+1\rangle\,\langle i-1, i\rangle + |i\rangle\,\langle i+1, i-1\rangle = 0$, and the identity follows upon using $p_i = |i\rangle^{\dot a}[i|^a$

Exercise Solution

Solution:

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\begin{aligned}
p_i &= y_i - y_{i+1} \\
&= \frac{|i\rangle^{\dot a}\,[\mu_{i-1}|^a - |i-1\rangle^{\dot a}\,[\mu_i|^a}{\langle i-1, i\rangle} - \frac{|i+1\rangle^{\dot a}\,[\mu_i|^a - |i\rangle^{\dot a}\,[\mu_{i+1}|^a}{\langle i, i+1\rangle}
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&= \frac{|i\rangle^{\dot a}\,\langle i, i+1\rangle\;[\mu_{i-1}|^a - \big(|i-1\rangle^{\dot a}\,\langle i, i+1\rangle\; + |i+1\rangle^{\dot a}\,\langle i-1, i\rangle\,\big)\,[\mu_i|^a + |i\rangle^{\dot a}\,\langle i-1, i\rangle\;[\mu_{i+1}|^a}{\langle i-1, i\rangle\langle i, i+1\rangle} \\
&= \frac{|i\rangle^{\dot a}\,\langle i, i+1\rangle\;[\mu_{i-1}|^a - \big(|i\rangle^{\dot a}\,\langle i+1, i-1\rangle\,\big)\,[\mu_i|^a + |i\rangle^{\dot a}\,\langle i-1, i\rangle\;[\mu_{i+1}|^a}{\langle i-1, i\rangle\langle i, i+1\rangle} \\
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&= \frac{|i\rangle^{\dot{a}}\,\langle i, i+1\rangle\;[\mu_{i-1}|^a - \left(|i\rangle^{\dot{a}}\,\langle i+1, i-1\rangle\,\right)[\mu_i|^a + |i\rangle^{\dot{a}}\,\langle i-1, i\rangle\;[\mu_{i+1}|^a}{\langle i-1, i\rangle\langle i, i+1\rangle} \\
&= \frac{|i\rangle^{\dot{a}}\Big(\,\langle i, i+1\rangle\;[\mu_{i-1}|^a - \langle i+1, i-1\rangle\;[\mu_i|^a + \langle i-1, i\rangle\;[\mu_{i+1}|^a\Big)}{\langle i-1, i\rangle\langle i, i+1\rangle}
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\end{aligned}
$$

$$
= \frac{|i\rangle^{\dot{a}} \Big(\langle i, i+1\rangle \; [\mu_{i-1}|^a - \langle i+1, i-1\rangle \; [\mu_i|^a + \langle i-1, i\rangle \; [\mu_{i+1}|^a \Big)}{\langle i-1, i\rangle \langle i, i+1\rangle}
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&= \frac{|i\rangle^{\dot a}\langle i, i+1\rangle\ [\mu_{i-1}|^a - \left(|i-1\rangle^{\dot a}\langle i, i+1\rangle\ + |i+1\rangle^{\dot a}\langle i-1, i\rangle\right)[\mu_i|^a + |i\rangle^{\dot a}\langle i-1, i\rangle\ [\mu_{i+1}|^a}{\langle i-1, i\rangle\langle i, i+1\rangle} \\
&= \frac{|i\rangle^{\dot a}\langle i, i+1\rangle\ [\mu_{i-1}|^a - \left(|i\rangle^{\dot a}\langle i+1, i-1\rangle\right)[\mu_i|^a + |i\rangle^{\dot a}\langle i-1, i\rangle\ [\mu_{i+1}|^a}{\langle i-1, i\rangle\langle i, i+1\rangle} \\
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where we used the Schouten identity $|i-1\rangle\langle i, i+1\rangle + |i+1\rangle\langle i-1, i\rangle + |i\rangle\langle i+1, i-1\rangle = 0$, and the identity follows upon using $p_i = |i\rangle^{\dot a}[i|^a$

Recap: Goal of Paper

The goal of the paper is to relate Grassmannian integral formulas written in 3 different formulations:

Twistor Space:

$$\tilde{\mathcal{L}}_{n;k}(\mathcal{W}) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{4\tilde{k}|4\tilde{k}}(B\cdot\mathcal{W})}{m_1 m_2 \ldots m_n}\,.$$

Momentum Space:

$$\mathcal{L}_{n;k}(\lambda,\tilde{\lambda},\tilde{\eta}) = \int \frac{d^{\tilde{k}\times n}B}{GL(\tilde{k})} \frac{\delta^{2\tilde{k}}(B_{\alpha i}\,\tilde{\lambda}_i)\,\delta^{2(n-\tilde{k})}(B^{\perp}_{\alpha i}\lambda_i)\,\delta^{(4\tilde{k})}(B_{\alpha i}\,\tilde{\eta}_{iA})}{m_1 m_2 \cdots m_n}\,,$$

Momentum Twistor Space:

$$\mathcal{L}_{n;k}(\mathcal{Z}) = \mathcal{A}_n^{\mathrm{MHV}} \int \frac{d^{k\times n}C}{GL(k)} \frac{\delta^{4k|4k}(C\cdot\mathcal{Z})}{M_1 M_2 \cdots M_n}\,.$$

Next Time

The plan for next time:

-Go through the calculations of how to transform from one Grassmann integral representation to another (big focus on momentum space to momentum twistor space version - related to T-duality)

Depending on time:

-One immediate application: can use this method to write the momentum twistor space ABJM Grassmannian integral formula (which was supposed to facilitate the jump to the ABJM amplituhedron)

-Do an example of an explicit calculation with one of these Grassmannian integral formulae