(2.1)

Consider the momentum vector

$$p^\mu = (E, Esin\theta cos\phi, Esin\theta sin\phi, Ecos\theta)$$

Express $p_{a\dot{b}}$ and $p^{\dot{a}b}$ in terms of $E, sin\frac{\theta}{2}, cos\frac{\theta}{2}, e^{\pm i\phi}$

Show that the helicity spinor $|p\rangle^{\dot{a}} = \sqrt{2E}\begin{pmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$ solves the massless Weyl equation. Find expressions for the spinors $\langle p|_{\dot{a}}, |p]_a, [p|^a$ and check that they satisfy $p_{a\dot{b}} = -|p]_a \langle p|_{\dot{b}}$ and $p^{\dot{a}b} = -|p\rangle^{\dot{a}}[p|^b$.

$$\begin{aligned}
p_{a\dot{b}} &= p_\mu(\sigma^\mu)_{a\dot{b}} \\
&= \begin{pmatrix} -p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix} \\
&= \begin{pmatrix} -E+Ecos\theta & Esin\theta cos\phi - iEsin\theta sin\phi \\ Esin\theta cos\phi + iEsin\theta sin\phi & -E-Ecos\theta \end{pmatrix} \\
&= E\begin{pmatrix} -1+cos\theta & sin\theta(cos\phi - isin\phi) \\ sin\theta(cos\phi+isin\phi) & -1-cos\theta \end{pmatrix} \\
&= E\begin{pmatrix} -1+cos\theta & sin\theta e^{-i\phi} \\ sin\theta e^{i\phi} & -1-cos\theta \end{pmatrix} \\
&= 2E\begin{pmatrix} -sin^2\frac{\theta}{2} & cos\frac{\theta}{2}sin\frac{\theta}{2}e^{-i\phi} \\ cos\frac{\theta}{2}sin\frac{\theta}{2}e^{i\phi} & -cos^2\frac{\theta}{2} \end{pmatrix}
\end{aligned}$$

where in the last step we used $cos2\theta = cos^2\theta sin^2\theta$ and $sin2\theta = 2cos\theta sin\theta$ and divided all thetas by 2.

Similarly,

$$
\begin{aligned}
p^{\dot{a}b} &= p_\mu(\overline{\sigma}^\mu)\dot{a}b \\
&= \begin{pmatrix} -p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & -p^0 + p^3 \end{pmatrix} \\
&= -\begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \\
&= -\begin{pmatrix} E + Ecos\theta & Esin\theta cos\phi - iEsin\theta sin\phi \\ Esin\theta cos\phi + iEsin\theta sin\phi & E - Ecos\theta \end{pmatrix} \\
&= -2E\begin{pmatrix} 1 + cos\theta & sin\theta(cos\phi - isin\phi) \\ sin\theta(cos\phi + isin\phi) & 1 - cos\theta \end{pmatrix} \\
&= -2E\begin{pmatrix} sin^2\frac{\theta}{2} & cos\frac{\theta}{2}sin\frac{\theta}{2}e^{-i\phi} \\ cos\frac{\theta}{2}sin\frac{\theta}{2}e^{i\phi} & cos^2\frac{\theta}{2} \end{pmatrix}
\end{aligned}
$$

...

(2.3)

Prove the Fierz identity (2.36)

$\langle 1|\gamma^\mu|2]\langle 2|\gamma_\mu|4] = 2\langle 13\rangle[24]$

I first wrote out:

$$
\begin{aligned}
\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu 4] &= [2|\gamma^\mu|1\rangle[4|\gamma_\mu|3\rangle \\
&= \langle 1|\overline{\sigma}^\mu|2]\langle 3|\overline{\sigma}_\mu|4] \\
&= \langle 1|_{\dot{a}}(\overline{\sigma}^\mu)^{\dot{a}b}|2]_b\langle 3|_{\dot{c}}(\overline{\sigma}^\mu)^{\dot{c}d}|4]_d
\end{aligned}
$$

Then realizing that I could use (2.33) $[k|\gamma^\mu|p\rangle = \langle p|\gamma^\mu|k]$ to make my life easier, I did:

$$\begin{aligned}
\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu 4] &= [2|\gamma^\mu|1\rangle[4|\gamma_\mu|3\rangle \text{ by (2.33)} \\
&= [2|^{\dot{a}}(\overline{\sigma}^\mu)_{\dot{a}b}|1\rangle^b[4|^{\dot{c}}(\overline{\sigma}_\mu)_{\dot{c}d}|3\rangle^d \\
&= [2|^{\dot{a}}|1\rangle^b[4|^{\dot{c}}|3\rangle^d((\overline{\sigma}^\mu)_{\dot{a}b}(\overline{\sigma}_\mu)_{\dot{c}d}) \\
&= [2|^{\dot{a}}|1\rangle^b[4|^{\dot{c}}|3\rangle^d(-2\epsilon_{ac}\epsilon_{\dot{b}\dot{d}}) \text{ by (A.5)} \\
&= -2[2|^a|4]_a|1\rangle^{\dot{b}}\langle 3|_{\dot{b}} \text{ by (A.14)} \\
&= 2\,\langle 13\rangle\,[24] \text{ by (2.22)}
\end{aligned}$$

Is this (2.26)?

In Gr(2,n), the form for the U(1) decoupling identity is a sum of form of permuted positroid (planar) forms, e.g. $A_n[153n2\cdots 6(n-1)]$ has form $\frac{1}{(15)(53)(3n)(n2)\cdots(6,n-1)(n-1,1)}$.

It will be easier for the proof that the U(1) decoupling identity follows from the Plucker/Schouten relations, to apply the shift to the n-th momentum:

$$A_n[1n234\cdots n{-}2,n{-}1]+A_n[12n34\cdots n{-}2,n{-}1]+\cdots A_n[1234\cdots n{-}2,n,n{-}1]+A_n[1n234\cdots n{-}2,n{-}1,n]$$

$$\begin{aligned}
&= \frac{1}{(1n)(n2)(23)\cdots(n-1,1)} + \frac{1}{(12)(2n)(n3)\cdots(n-1,1)} + \cdots + \frac{1}{(12)(23)\cdots(n-2,n)(n,n-1)(n-1,1)} \\
&\quad + \frac{1}{(12)(23)\cdots(n-2,n-1)(n-1,n)(n1)} \\
&= \sum_{i=1}^{n-1} \frac{1}{(12)(23)(34)\cdots(in)(n,i+1)\cdots(n-1,1)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{-(12)(n3)\cdots(n,n-1)-(23)(n1)(n4)\cdots(n,n-1)-(34)(n1)(n2)(n5)\cdots(n,n-1)-\cdots}{(12)(23)\cdots(n-1,1)(n1)(n2)\cdots(n,n-1)} \\
&\quad \frac{-(n-2,n-1)(n1)(n2)\cdots(n,n-3)+(n-1,1)(n2)(n3)\cdots(n,n-2)}{(12)(23)\cdots(n-1,1)(n1)(n2)\cdots(n,n-1)}
\end{aligned}$$

Notice that the denominator contains two pieces: one that is just the cyclic minors $(12)(23)\cdots(n-1,1)$, and the other that consists of n with all the other momenta: $(n1)(n2)(n3)\cdots(n,n-1)$. Each one has $n-1$ terms so the denominator has $2n-2$ terms in total. Also notice that for the numerator, first of all, all terms are negative except the last one, which is positive. And for each i-th numerator term, it contains the later piece $(n1)(n2)(n3)\cdots(n,n-1)$, except that $(ni)(n,i+1)$ are dropped and replaced by $(i,i+1)$.

We want to show that this vanishes, more precisely, the numerator vanishes, using the Plucker relations. The proof goes via induction. We first prove the base case for $n=4$, then we show that if this is true for any n, then it is also true for $n+1$. In other words, we assume the vanishing of the numerator in the previous equation, (let's call this Equation (A)) and we want to show using (A) (and the Plucker relations), that:

$$= \frac{-(12)(n+1,3)\cdots(n+1,n)-(23)(n+1,1)(n+1,4)\cdots(n+1,n)-(34)(n+1,1)(n+1,2)(n+1,5)\cdots(n+1}{(12)(23)\cdots(n-1,n)(n1)(n+1,1)(n+1,2)\cdots(n+1,n)}$$

$$\frac{-(n-1,n)(n+1,1)(n+1,2)\cdots(n+1,n-2)+(n,1)(n+1,2)(n+1,3)\cdots(n+1,n-1)}{(12)(23)\cdots(n-1,n)(n1)(n+1,1)(n+1,2)\cdots(n+1,n)}$$

We take the last two terms of this numerator:

$-(n-1,n)(n+1,1)(n+1,2)\cdots(n+1,n-2)+(n,1)(n+1,2)(n+1,3)\cdots(n+1,n-1)$

$=(n+1,2)\cdots(n+1,n-2)[-(n-1,n)(n+1,1)+(n1)(n+1,n-1)]$

$=(n+1,2)\cdots(n+1,n-2)[(1,n-1)(n,n+1)]$ by applying Plucker to the thing in square brackets.

Thus, the numerator is now: $-(12)(n+1,3)\cdots(n+1,n)-(23)(n+1,1)(n+1,4)\cdots(n+1,n)+\cdots+(n+1,2)\cdots(n+1,n-2)(1,n-1)(n,n+1)$.

We can factor out the common $(n+1,n)$:

$(n+1,n)*[-(12)(n+1,3)\cdots(n+1,n-1)-(23)(n+1,1)(n+1,4)\cdots(n+1,n-1)+\cdots-(n+1,2)\cdots(n+1,n-2)(1,n-1)]$

And this is precisely equation (A) (with momentum $n+1$ instead of momentum n) so it vanishes by the induction hypothesis.

We didn't show the base case but this is easy:

$\frac{1}{(12)(23)(34)(41)} + \frac{1}{(42)(21)(13)(34)} + \frac{1}{(42)(23)(31)(14)}$

$= \frac{(42)(13)-(23)(41)-(12)(34)}{(12)(23)(34)(41)(42)(13)} = 0$ by the Plucker relations.

(2.29)

Start with $A_5[a, \{2,3\}, 5, \{4\}]$ to show that the Kleiss-Kuijf relation gives: $A_5[12345]+A_5[12354]+A_5[12435]+A_5[14235] = 0$ (2.88). Then show that (2.88) together with the U(1) decoupling relation implies that (2.87).

From here on out, I will write just A instead of A_5 since there is no ambiguity. Note that (2.87) says: $A[12534] = A[12435] + A[14235] + A[14325]$.

The equation (2.88) is a direct writing of the KK relations. First, I realized that two of the terms are already the same: A[14325] and A[12435]. In order to keep the number of terms smaller so that I'm not dealing with huge equations and making lots of mistakes, I saw that there was no way to apply U(1) to (2.87) and have things cancel, but I could do this with the two terms A[12345] and A[12354] of (2.88) rather easily. I can get this cancellation either by moving the 4 or the 5 with U(1), but because the expression we are dealing with have a bunch of A[14...], I want as many of these as possible, so I moved the 4 in U(1):

A[12354]+A[14235]+A[12435]+A[12345]=0. Plugging this into (2.88), we not only get cancellation of the A[12345] but also cancellation of the A[14235] (which I wasn't expecting and didn't want because it shows up in the final expression). The result is:

A[12435]-A[12435]=0.

Actually, you don't need to apply U(1) to (2.88) to get (2.87): since they are both U(1) equations. It turns out that the U(1) that produces (2.87) is applying the U(1) shift to 2 in A[12435]:

A[12435]+A[14235]+A[14325]+A[14352]=0. Use reflection on A[14352] to get -A[12534]. Then this is (2.87).

So this solution could have been written very compactly:

Apply U(1)-decoupling to momentum 4 in $A[12345]$. This produces: $A[12354] + A[14235] + A[12435] + A[12345] = 0$, which is just (2.88) itself. Now, apply U(1) to momentum 2 in $A[12435]$: $A[12435] + A[14235] + A[14325] + A[14352] = 0$. Finally, using reflection on $A[14352]$ to get $-A[12534]$. This is (2.87).

(3.2)

Convince yourself that for large z the amplitude (3.11) falls off as $1/z$ under a $[-,-\rangle$ shift (i.e. choose i and j to be the two negative helicity lines). What happens under the three other types of shifts? Note the difference between shifting adjacent/non-adjacent lines.

The Parke-Taylor formula is $A_n[1^-2^-3^+\cdots n^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$

We have to deal with 2 cases: adjacent and nonadjacent.

For the adjacent case, let's do $A_n[\hat{1}^-\hat{2}^-3^+\cdots n^+]$

The shift is as usual $[1 \to [\hat{1} = [1 + z[2$, and $2\rangle \to \hat{2}\rangle = 2\rangle - z1\rangle$.

Thus,

$$\begin{aligned} A_n[\hat{1}^-\hat{2}^-3^+\cdots n^+] &= \frac{[\langle 1(2\rangle - z1\rangle)]^4}{\langle 1(2\rangle - z1\rangle)(\langle 2 - z\langle 1)3\rangle\,\langle 34\rangle\cdots\langle n1\rangle} \\ &= \frac{[\langle 12\rangle - z\,\langle 11\rangle]^4}{(\langle 12\rangle - z\,\langle 11\rangle)(\langle 23\rangle - z\,\langle 13\rangle)\,\langle 34\rangle\cdots\langle n1\rangle} \\ &= \frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\cdots\langle n1\rangle - z\,\langle 12\rangle\,\langle 13\rangle\,\langle 34\rangle} \end{aligned}$$

and since as $z \to \infty$ the term with z dominates, this approaches

$$-\frac{1}{z}\frac{\langle 23\rangle}{\langle 13\rangle}\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\cdots\langle n1\rangle}$$

So it goes to zero as $z \to \infty$

For the case of the particles being nonadjacent, the amplitude goes as $1/z^2$ as $z \to \infty$.

For:

$[-,+\rangle$ adjacent: $1/z$ $[-,+\rangle$ nonadjacent:$1/z^2$

$[+,-\rangle$ adjacent: z^3 $[+,-\rangle$ nonadjacent: z^2

$[+,+\rangle$ adjacent: $1/z$ $[+,+\rangle$ nonadjacent: $1/z^2$

The full calculations are on pages 176-179 of my notebook.