Recall from last time:

fiber bundle, vector bundle, sections, canonical line bundle of $\mathbb{RP}^n$ or $\mathbb{CP}^n$

fibers are vector spaces

For the rest, assume base space is paracompact Hausdorff

Def A Hausdorff space X is paracompact if for each open cover $\{U_\alpha\}$ of X, there is a partition of unity $\{\phi_\beta\}$ subordinate to the cover, meaning

i) $\phi_\beta : X \to [0,1]$

ii) $\operatorname{supp} \phi_\beta \subseteq$ some U_α

iii) $\forall x \in X$, $\exists\, U \ni x$ s.t. $\phi_\beta|_U \neq 0$ for finite # of β

iv) $\sum_\beta \phi_\beta(x) = 1 \quad \forall x \in X$

Examples of paracompact spaces:

- compact Hausdorff spaces, CW complexes, metric spaces, unions of increasing sequences $X_1 \subseteq X_2 \subseteq \cdots$ of compact Hausdorff spaces with the weak/direct limit topology (U open $\iff U \cap X_i$ is open $\forall i$)

Important Notion: Isomorphic Vector Bundles

Def An <u>isomorphism between vector bundles</u> $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ over the same base space B is a homeomorphism $h : E_1 \to E_2$ taking each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism

Complex Vector Bundle

$\pi: E \to B$ where each fiber $\pi^{-1}(b)$ has the structure of a complex vector space, with trivializations $\pi^{-1}(U) \to U \times \mathbb{C}^n$ that are homeomorphisms which are complex linear maps (isomorphisms?) on each fiber

(a holomorphic vector bundle has the much stronger condition that the trivializations be biholomorphic

? but I remember vaguely that a bijective holomorphic map automatically has holomorphic inverse?)

If we have a complex structure on a $2n$-dim vector bundle:

$J: E \to E$ which maps each fiber $\mathbb{R}$-linearly to itself and which satisfies

$J(J(v)) = -v$ for every vector in E,

we can make each fiber F_b into a complex vector space by setting

$(x+iy)v = xv + J(yv)$

We can verify the local triviality to show that E becomes a complex vector bundle

~~Important goal: we want to know when vector bundles are isomorphic!~~

Is a vector bundle isomorphic to the trivial bundle?

Assuming the vector bundle is of dim ≥ 1, a necessary condition is that it has a nonvanishing vector field.

But not sufficient for dim ≥ 1.

Lemma An n-dim bundle $p: E \to B$ is isomorphic to the trivial bundle iff it has ~~[illegible]~~ a basis of sections (that is, n sections $s_1, \dots, s_n$ s.t. $s_1(b), \dots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$)

Proof ($\Rightarrow$) Suppose E is isomorphic to the trivial bundle $B \times \mathbb{R}^n$. We already have a basis of sections for $B \times \mathbb{R}^n$ by taking a basis $s_1, \dots, s_n$ for $\mathbb{R}^n$, and letting the sections be (b, s_i) for all $b \in B$. Then since by definition an isomorphism of vector bundles is a linear isomorphism on each fiber, the map from $E|_b$ to $b \times \mathbb{R}^n$ is given by an element $G(b) \in GL(n, \mathbb{R})$ for each b, which for $G(b)^{-1}$ takes linearly independent elements of $\mathbb{R}^n$ to linearly independent elements of $E|_b$.

($\Leftarrow$) Suppose we have n linearly independent sections s_i, we can form the map

$h: B \times \mathbb{R}^n \to E$ is a linear isomorphism on each fiber

$(b, t_1, \dots, t_n) \mapsto \sum_i t_i s_i(b)$ (change of basis from $(1,0,0,\dots,0)$ to $(s_1(b), 0, \dots, 0)$)?

h is continuous since for any $b \in B$ with trivialization α on $U \ni b$, we have

$$U \times \mathbb{R}^n \xrightarrow{h} E|_U \xrightarrow{\alpha} U \times \mathbb{R}^n$$

$$(b, t_1, \dots, t_n) \longmapsto (b, s_1 t_1, \dots, s_n t_n)$$

The preimage under $(\alpha \circ h)^{-1}$ of an open set here is an open set in $U \times \mathbb{R}^n$ $(b, t_1, \dots, t_n)$.

Thus since α and $\alpha \circ h$ are continuous, so is h.

To complete the proof, we need to know that h is a homeomorphism. Since the linear isomorphism implies that it is bijective, we just need to show that h^{-1} is continuous (so h is bicontinuous). This follows by looking at the map ~~transition functions~~ on a local trivialization for $U \times \mathbb{R}^n \to U \times \mathbb{R}^n$. Since they are given by elements of $GL(n, \mathbb{R})$ varying continuously with $b \in B$, their inverses, also elements of $GL(n, \mathbb{R})$, also vary continuously with $b \in B$.

We can form direct sums and tensor products of vector bundles:

For $p_1: E_1 \to B$, $p_2: E_2 \to B$,

$$E_1 \oplus E_2 := \{ (v_1, v_2) \in E_1 \times E_2 \mid p_1(v_1) = p_2(v_2) \}$$

$$E_1 \otimes E_2 := \left\{ \coprod_{x \in B} p_1^{-1}(x) \otimes p_2^{-1}(x) \right\}$$ with topology on $p_1^{-1}(U) \otimes p_2^{-1}(U)$

by letting the fiberwise tensor product map

$h_1 \otimes h_2 : p_1^{-1}(U) \otimes p_2^{-1}(U) \to U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ be a homeomorphism

where $h_1 : p_1^{-1}(U) \to U \times \mathbb{R}^{n_1}$, $h_2 : p_2^{-1}(U) \to U \times \mathbb{R}^{n_2}$ are trivializations

(a little more work is required to show that this works)

[For trivialization on $\{U_\alpha\}$, we can also look at this in terms of transition functions

$g^1_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_{n_1}(\mathbb{R})$, $g^2_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_{n_2}(\mathbb{R})$

Since the transition functions for $E_1 \otimes E_2$ are given by $g^1_{\beta\alpha} \otimes g^2_{\beta\alpha}$]

Note: For each vector bundle $E \to B$ with B compact Hausdorff, there exists a vector bundle $E' \to B$ such that $E \oplus E'$ is the trivial bundle.

(proof on p13 Hatcher)

This can fail if B is noncompact.

Note that E' might not be trivial. If E' is trivial, then we call $E \to B$ stably trivial.

Examples: the tangent bundle to S^n is stably trivial (take direct sum with normal bundle)

the canonical line bundle of $\mathbb{RP}^n$ ~~is stably~~ becomes trivial (not stably trivial in general)

(by taking direct sum with the orthogonal complement)

(Issue: the Chern, Stiefel-Whitney, and Pontryagin classes are stable invariants, meaning they do not change upon taking the direct sum with a trivial bundle, so they cannot distinguish between the tangent bundles to spheres and trivial bundles $S^n \times \mathbb{R}^n$

(only Euler class is not))

For S^n we know that the tangent bundle has a nonvanishing section iff n is odd (proof using degrees - p135 Hatcher). So for S^n to be trivial, n cannot be even.

So when does TP have a basis of nonvanishing sections? When $n = 0, 1, 3, 7$:

related to division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, (i.e. when is TP parallelizable)

(can prove with elements of K-theory) ~~I believe these are also the only spheres that are Lie groups.~~ (p179 Lee SM)

Since Lie groups are parallelizable, these are the only possibilities: turns out only S^1 and S^3 have Lie group structure let $\mathrm{Vect}^n(B)$ denote the set of isomorphism classes of n-dimensional real vector bundles

$\mathrm{Vect}^n_{\mathbb{C}}(B)$ " complex "

For vector bundle $p: E \to B$, a map $f: A \to B$ gives us a function $f^*: \mathrm{Vect}(B) \to \mathrm{Vect}(A)$, called the pull back bundle

We visualize this with the commutative diagram:

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

This is of course a fiber product diagram with

$$E' = \{ (a,v) \in A \times E \mid f(a) = p(v) \}$$

The proposition (which we will not prove) states:

Given a map $f: A \to B$ and a vector bundle $p: E \to B$, there exists a vector bundle $p': E' \to A$ with a map $f': E' \to E$ taking the fiber of E' over each point $a \in A$ isomorphically onto the fiber of E over $f(a)$, and such a vector bundle E' is unique up to isomorphism.

Note that if we had an isomorphism $\begin{tikzcd} E \arrow[r, "g"] \arrow[dr, "p"'] & F \arrow[d, "q"] \\ & B \end{tikzcd}$, then we would have $g \circ f': E' \to F$

and we formed the pull back

$$\begin{tikzcd} E'' \arrow[r, "f''"] \arrow[d, "p''"'] & F \arrow[d, "q"] & E \arrow[l, "g"'] \arrow[dl, "p"] & E' \arrow[l, "f'"'] \arrow[d, "p'"] \\ A \arrow[r, "f"] & B & & A \arrow[ll, "f"'] \end{tikzcd}$$

so $E'' \cong E'$ via $(f')^{-1} \circ g^{-1} \circ f''$

That is, the iso class of E' does indeed only depend on the isomorphism class of E: changing to a different element in the isomorphism class containing E and apply f^* gives us another element of the isomorphism class of E'

Basic properties of pullbacks:

i) $(fg)^*(E) \cong g^*(f^*E)$

ii) $\mathbb{1}^* E \cong E$

iii) $f^*(E_1 \oplus E_2) \cong f^*E_1 \oplus f^*E_2$

iv) $f^*(E_1 \otimes E_2) \cong f^*E_1 \otimes f^*E_2$

Thm (For A paracompact) Given a vector bundle $p: E \to B$ and homotopic maps $f_0, f_1 : A \to B$, the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic

Proof uses the following prop: The restrictions of a vector bundle $E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is paracompact

↳ proof?

Corollary A homotopy equivalence $f: A \to B$ of paracompact spaces induces a bijection $f^*: \mathrm{Vect}^n(B) \to \mathrm{Vect}^n(A)$. In particular, every vector bundle over a contractible paracompact base is trivial. not true for holomorphic VB over algebraic varieties

Universal Bundle

(The idea is that every n-dim vector bundle $p: E \to B$, with B paracompact, can be expressed as the pullback of a single n-dim vector bundle $E_n \to G_n$)

We define G_n as $G_n = G_n(\mathbb{R}^\infty) = \bigcup_k G_n(\mathbb{R}^k)$

We define $G_n(\mathbb{R}^k)$ as the Grassman manifold of all n-planes in $\mathbb{R}^k$, $n \leq k$

We give it a topology using the Stiefel manifold $V_n(\mathbb{R}^k)$, the space of orthonormal n-frames in $\mathbb{R}^k$: this is a subspace of $\underbrace{S^{k-1} \times \cdots \times S^{k-1}}_{n \text{ copies}}$

need Hermitian inner product / easier: for AG: nonorthogonal.

which is a compact set. Since orthogonality is an algebraic condition (dot product zero), it is a closed, so since a closed subspace of a compact set is compact, $V_n(\mathbb{R}^k)$ is compact.

And since we give $G_n(\mathbb{R}^k)$ the quotient topology with respect to the surjection $V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ sending each element to the n-space it spans, $G_n(\mathbb{R}^k)$ is compact as well.

Now coming back to $G_n = G_n(\mathbb{R}^\infty) = \bigcup_k G_n(\mathbb{R}^k)$, the inclusions $G_n(\mathbb{R}^k) \subseteq G_n(\mathbb{R}^{k+1}) \subseteq \cdots$ are induced by the inclusions $\mathbb{R}^k \subseteq \mathbb{R}^{k+1} \subseteq \cdots$ and $G_n(\mathbb{R}^\infty)$ has the weak/direct limit topology, meaning $G_n(\mathbb{R}^\infty)$ is open iff it intersects each $G_n(\mathbb{R}^k)$ in an open set.

Note that $G_n(\mathbb{R}^\infty)$ is the set of all n-dim vector subspaces of the vector space $\mathbb{R}^\infty$

We have the canonical bundle $E_n(\mathbb{R}^k)$ over $G_n(\mathbb{R}^k)$ defined as:

$$E_n(\mathbb{R}^k) = \{(\ell, v) \in G_n(\mathbb{R}^k) \times \mathbb{R}^k \mid v \in \ell\}$$

This gives us a canonical bundle $E_n = E_n(\mathbb{R}^\infty)$ over $G_n = G_n(\mathbb{R}^\infty)$ since the inclusions $\mathbb{R}^k \subseteq \mathbb{R}^{k+1} \subseteq \cdots$ give inclusions $E_n(\mathbb{R}^k) \subset E_n(\mathbb{R}^{k+1}) \subseteq \cdots$ and we let $E_n(\mathbb{R}^\infty) = \bigcup_k E_n(\mathbb{R}^k)$ with the weak topology

Prop The projection $p: E_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$, $p(l,v) = l$ is a vector bundle

Proof Idea. Suppose k is finite (we can take $k=\infty$ at the end)

For $l \in G_n(\mathbb{R}^k)$, let $\pi_l : \mathbb{R}^k \to l$ be the orthogonal projection, and let

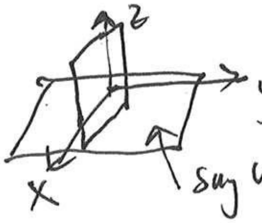
$U_l = \{ l' \in G_n(\mathbb{R}^k) \mid \pi_l(l') \text{ has dimension } n \}$.

We prove the proposition by showing that U_l is open in $G_n(\mathbb{R}^k)$ and that the map

$$h: p^{-1}(U_l) \to U_l \times l \cong U_l \times \mathbb{R}^n$$
$$(l', v) \mapsto (l', \pi_l(v))$$ is a local trivialization of $E_n(\mathbb{R}^k)$.

(The set U_l cover $G_n(\mathbb{R}^k)$)

Picture:



Say we have the element given by the x-y plane in $\mathbb{R}^3$. Then U_l consists of all planes except the ones containing the z-axis.
It's clear that if we spin the x-z plane drawn above around the z-axis, we get something like S^1, a codimension-1 set, so U_l is open.
We can actually think of this in terms of the normals, so then the space is an $\mathbb{R}^2$ bundle over $\mathbb{RP}^2$:

this is the vector fiber over the x-y plane

It's intuitively clear that the tangent to any point, except the points along the boundary, are at an angle, so can be projected isomorphically onto E_0.

Thus $p^{-1}(U_l) = \underset{\text{tangent}}{T}(\mathbb{RP}^2 \setminus \partial\mathbb{RP}^2) \cong (\mathbb{RP}^2 \setminus \partial \mathbb{RP}^2) \times E_0$

$\cong U_l \times \mathbb{R}^2$

Note: $[X,Y]$ denotes the homotopy classes of maps $f: X \to Y$

The following theorem explains why G_n is called the <u>classifying space</u> for n-dim vector bundles and $E_n \to G_n$ is ~~termed~~ called the <u>universal bundle</u> (for n-dim vector bundles)

~~Thm~~ For paracompact X, the map $[X, G_n] \to \mathrm{Vect}^n(X)$ $\big([X, G_n(\mathbb{C}^\infty)] \leftrightarrow \mathrm{Vect}^n_{\mathbb{C}}(X)\big)$ is a bijection

$$[f] \mapsto f^* E_n$$

That is, if two maps $f, g: X \to G_n$ are homotopic, then their pullback bundles of the universal bundle are isomorphic as vector bundles.

(If X and Y are homotopy equivalent spaces, f and g homotopy inverses ?

$$\begin{tikzcd} & E_n \arrow[d] \\ X \arrow[r, "h_X"] \arrow[d, shift left, "f"] & G_n \\ Y \arrow[ur, "h_Y"'] \arrow[u, shift left, "g"] & \end{tikzcd}$$

<u>Proof</u> To prove the theorem, we have to show bijectivity (which I will not do). The key thing ~~Then we note that~~ this uses is that: given

$$\begin{array}{c} E \cong f^*(E_n) \\ \downarrow p \\ X \end{array} \quad \overset{\text{equivalent}}{\Longleftrightarrow} \quad g: E \to \mathbb{R}^\infty \text{ that is a linear injection on each fiber.}$$

($\Rightarrow$) We have a diagram

$$\begin{tikzcd} E \overset{h}{\cong} f^*(E_n) \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, ""] & E_n \arrow[r, "\pi"] \arrow[d] & \mathbb{R}^\infty \\ X \arrow[r, "f"] & G_n & \end{tikzcd}$$

where $\pi(\ell, v) = v$ is a linear injection on each fiber, so $g = \pi \circ \tilde{f} \circ h: E \to \mathbb{R}^\infty$ since all of the composite maps have this property

($\Leftarrow$) Given a map $g: E \to \mathbb{R}^\infty$ that is a linear injection on each fiber, define $f: X \to G_n$, $x \mapsto g(p^{-1}(x)) \in n$-plane

so as

$$\begin{tikzcd} E \arrow[r, "g \circ h"] \arrow[d] & E_n \arrow[d] \\ X \arrow[r, "f"] & G_n \end{tikzcd} \qquad \begin{tikzcd} p^{-1}(x) \arrow[r, "(f \circ h)^{-1}"] \arrow[d] & g(p^{-1}(x)) \leftarrow \text{plane} \arrow[d] \\ x \arrow[r, mapsto, "f"] & g(p^{-1}(x)) \leftarrow \text{point} \end{tikzcd}$$

To prove injectivity, we use a result that two maps of X to G_n such that their pullbacks are isomorphic, when composed with the projection π to form two maps $g_1, g_2: E \to \mathbb{R}^\infty$ as above, have the property that g_0 and g_1 are homotopic through maps g_t that are linear injections on fibers.

Example: Tangent bundle to S^n:

$$\begin{array}{c} E \\ \downarrow p \\ S^n \end{array} \quad \text{with } E = \{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid x \perp v\}$$

(which we consider as based at the origin), so Each fiber $p^{-1}(x)$ is an n-plane in $\mathbb{R}^{n+1}$ is therefore a point in $G_n(\mathbb{R}^{n+1})$. Thus, we have a map $S^n \to G_n(\mathbb{R}^{n+1})$, $x \mapsto p^{-1}(x)$

Via the inclusion $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^\infty$, we can view this as a map $f: S^n \to G_n(\mathbb{R}^\infty) = G_n$, and E is exactly the pullback $f^*(E_n)$

The topology of Grassmannians

$G_n(\mathbb{R}^k)$ is a closed ~~(we already showed)~~ manifold of dim $n(k-n)$

$G_n(\mathbb{C}^k)$ " " dim $2n(k-n)$

And, $G_n(\mathbb{R}^\infty)$ has the structure of a CW complex with each $G_n(\mathbb{R}^k)$ a finite subcomplex.

First, we note that $G_n(\mathbb{R}^k)$ is Hausdorff:

Proof For any $\ell, \ell' \in G_n(\mathbb{R}^k)$, we have a continuous function $f: G_n(\mathbb{R}^k) \to \mathbb{R}$ that takes different values on ℓ and ℓ'. Let $f = f_v$ for $v \in \ell$.

$f_v(\ell')$ is the length of v projected onto ℓ' orthogonally, so clearly

$f_v(\ell) = |v| > f_v(\ell')$

We describe the CW structure by taking a point in $G_n(\mathbb{R}^k)$.

This point can be denoted by a basis of n vectors spanning the n-plane.

This point is associated with a Schubert symbol $(\sigma_1, \dots, \sigma_n)$, which is defined as the numbers where the projections $p_i : \mathbb{R}^k \to \mathbb{R}^{k-i}$, $i: 0 \to k$ decreases (equivalently, the i's where the dimension of the kernel of p_i increases).

The Schubert symbol only depends on the point in $G_n(\mathbb{R}^k)$, not on any of the vectors spanning the plane, since any choice of such vectors can be transformed into another set by an element of $GL(n, \mathbb{R})$, and such transformations do not change the row space. (?) proof?

In particular, elementary row operations, which are in $GL(n, \mathbb{R})$ can be used to put the $n \times k$ matrix into a row echelon form.

For example, for $G_4(\mathbb{R}^{10})$, we could have something like:

$$\begin{bmatrix} x & x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & x & 0 & x & 1 & 0 & 0 & 0 & 0 & 0 \\ x & x & 0 & x & 0 & 1 & 0 & 0 & 0 & 0 \\ x & x & 0 & x & 0 & 0 & x & x & 1 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \sigma_1 - 1 = 3-1 = 2 \text{ stars} \\ \leftarrow \sigma_2 - 2 = 5-2 = 3 \text{ stars} \\ \leftarrow \sigma_3 - 3 = 6-3 = 3 \text{ stars} \\ \leftarrow \sigma_4 - 4 = 9-4 = 5 \text{ stars} \\ \hline 13 \end{array}$$

$\sigma = (3, 5, 6, 9)$

$13 \leftarrow$ so this point of $G_4(\mathbb{R}^{10})$ matrix defining the Schubert symbol $(3,5,6,9) = \sigma$ gives us a set $e(\sigma)$ of all n-planes having σ as their Schubert symbol, where here $e(\sigma) \cong \mathbb{R}^{13}$

Thus, $\dim e(\sigma) = (\sigma_1 - 1) + \dots + (\sigma_n - n)$ for the real case

$\dim e(\sigma) = (2\sigma_1 - 2) + \dots + (2\sigma_n - 2n)$ for the complex case

Connection between σ and the Schubert varieties stuff?

Proposition The cells $e(\sigma)$ are the cells of a CW structure on $G_n(\mathbb{R}^k)$

Ex $G_2(\mathbb{R}^4)$ has $\binom{k}{n} = \binom{4}{2} = 6$ cells for the 6 distinct Schubert symbols:

$$(1,2) \qquad (1,3) \qquad (1,4) \qquad (2,3) \qquad (2,4) \qquad (3,4)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \underset{0}{\underline{\begin{matrix} \text{dim.} \\ 1-1=0 \\ 2-2=0 \end{matrix}}} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix} \underset{1}{\underline{\begin{matrix} 1-1=0 \\ 3-2=1 \end{matrix}}} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix} \underset{2}{\underline{\begin{matrix} 1-1=0 \\ 4-2=2 \end{matrix}}} \quad \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{bmatrix} \underset{2}{\underline{\begin{matrix} 2-1=1 \\ 3-2=1 \end{matrix}}} \quad \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix} \underset{3}{\underline{\begin{matrix} 2-1=1 \\ 4-2=2 \end{matrix}}} \quad \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix} \underset{4}{\underline{\begin{matrix} 3-1=2 \\ 4-2=2 \end{matrix}}}$$

The inclusions $G_n(\mathbb{R}^k) \subseteq G_n(\mathbb{R}^{k+1})$, which are inclusions of subcomplexes,
give us a CW structure on $G_n(\mathbb{R}^\infty)$.
We then get a CW complex structure on $G_\infty = G_\infty(\mathbb{R}^\infty) = \bigcup_n G_n(\mathbb{R}^\infty)$
via the natural inclusion $i: G_n \hookrightarrow G_{n+1}$ defined by
$i(\ell) = \mathbb{R} \times j(\ell)$ where $j: \mathbb{R}^\infty \to \mathbb{R}^\infty$ is the embedding $j(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$.

The dimension of $G_n(\mathbb{R}^k)$ is equal to the dimension of its highest dimension cell.
This occurs for the Schubert symbol $\sigma = (k-n+1, k-n+2, \dots, k)$ and has dimension

$$[(k-n+1)-1] + [(k-n+2)-2] + \dots + [k-n] = n \cdot (k-n) \text{ for } \mathbb{R}^k$$

$$[2(k-n+1)-2] + [2(k-n+2)-4] + \dots + 2k-2n = n \cdot 2(k-n) = 2n(k-n) \text{ for } \mathbb{C}^k$$

Thus, $G_n(\mathbb{R}^k)$ is a $n(k-n)$-dim manifold near points in the interior of this cell,
but $G_n(\mathbb{R}^k)$ is homogeneous by $GL(k, \mathbb{R})$ which takes planes to planes,
(what if the planes are orthogonal?)
so $G_n(\mathbb{R}^k)$ is a manifold, which is closed by what we showed above.

Characteristic Classes

Characteristic classes are cohomology classes associated to vector bundles.

We saw before that an n-dim vector bundle $E \to B$ being trivial is equivalent to its classifying map $f: B \to G_n$ being nullhomotopic.

This is difficult to determine.

Easier: two homotopic maps are equal on cohomology, and a nullhomotopic map is zero on cohomology, so a necessary condition for triviality of the bundle $E \xrightarrow{p} B$ is that its classifying map $f: B \to G_n$ induces the zero map on cohomology:

$f^* = 0 : H^n(G_n, R) \to H^n(B, R)$

(This is necessary but not sufficient for triviality, as we saw with spheres, which are stable invariants that most of the characteristic classes cannot detect)

Chern Classes (defined for complex vector bundles)

~~Theorem~~ There is a unique sequence of functions $c_1, c_2, \dots$ assigning to each complex vector bundle $E \to B$ a class $c_i(E) \in H^{2i}(B; \mathbb{Z})$, depending only on the isomorphism type of E, such that

a) $c_i(f^*(E)) = f^*(c_i(E))$ for a pullback $f^*(E)$

b) $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ for $c = 1 + c_1 + c_2 + \dots \in H^*(B, \mathbb{Z})$

c) $c_i(E) = 0$ if $i > \dim E$

d) For the canonical line bundle $E \to \mathbb{CP}^\infty$, $c_1(E)$ is a generator of $H^2(\mathbb{CP}^\infty, \mathbb{Z})$ specified in advance

Note: The sum $c(E) = 1 + c_1(E) + c_2(E) + \dots$ is the total Chern class.

c) implies that the total chern class of a vector bundle has only finitely many terms (so this sum does indeed live in $H^*(B; \mathbb{Z})$, the direct sum of the groups $H^i(B; \mathbb{Z})$)

need direct product if c was infinite

From the formal identity

$(1 + c_1 + c_2 + \dots)(1 + c_1' + c_2' + \dots) = 1 + (c_1 + c_1') + (c_2 + c_1 c_1' + c_2') + \dots$, it follows that the formula $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ is just a compact way of writing the relations

$$c_n(E_1 \oplus E_2) = \sum_{i+j=n} c_i(E_1) \smile c_j(E_2) \text{ where } c_0 = 1$$ (Whitney sum formula)

$E^{\perp}$ iso $\to$ $\mathbb{CP}^1$ — hyperplane

↑ generator of $\mathbb{Z}$

d) is a normalization and a nontriviality condition

otherwise we could set $c_i(E) = 0 \;\forall E, \forall i > 0$ to satisfy a)–c)

since for a fixed integer k, replacing $c_i \mapsto k^i c_i$ gives new functions satisfying a)–c)

$c_1(E) = -$ class of a hyperplane

dual to points

Proof The proof will use the Leray-Hirsch theorem describing the cohomology of certain fiber bundles.

<u>Leray-Hirsch Thm</u>

Let $F \to E \xrightarrow{p} B$ be a fiber bundle such that, for some commutative coefficient ring R:

a) $H^n(F;R)$ is a f.g. free R-module for each n (no torsion)

b) There exist classes $c_j \in H^{k_j}(E;R)$ whose restrictions $i^*(c_j)$ form a basis for $H^*(F;R)$ in each fiber F, where $i: F \to E$ is the inclusion

Then the map $\Phi: H^*(B;R) \otimes_R H^*(F;R) \to H^*(E;R)$ is an isomorphism

$$\sum_{ij} b_i \otimes i^*(c_j) \longmapsto \sum_{ij} p^*(b_i) \smile c_j$$

In other words, $H^*(E;R)$ is a free $H^*(B;R)$-module with basis $\{c_j\}$, where we view $H^*(E;R)$ as a module over the ring $H^*(B;R)$ by defining scalar multiplication by $bc = p^*(b) \smile c$ for $b \in H^*(B;R)$ and $c \in H^*(E;R)$

For vector bundle $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ with fiber $\mathbb{C}^n$, we have associated projective bundle $\begin{matrix} \mathbb{P}(E) \\ \downarrow \mathbb{P}(\pi) \\ B \end{matrix}$

where $\mathbb{P}(E)$ is the space of all lines thru the origin in all the fibers of E, and $\mathbb{P}(\pi)$ is the natural projection sending each line in $\pi^{-1}(b)$ to $b \in B$.

(we topologize $\mathbb{P}(E)$ as the quotient of the complement of the zero section of E, the quotient given by factoring out scalar multiplication on each fiber)

On a nbhd U in B where $E|_U \cong U \times \mathbb{C}^n$, the quotient is $U \times \mathbb{C}P^{n-1}$, so $\mathbb{P}(E)$ is a fiber bundle over B with fiber $\mathbb{C}P^{n-1}$.

To apply the Leray–Hirsch theorem for cohomology with $\mathbb{Z}$ coefficients to this bundle $\mathbb{P}(E) \to B$, we need classes $x_i \in H^{2i}(\mathbb{P}(E), \mathbb{Z})$ that restrict to the generators of $H^{2i}(\mathbb{C}P^{n-1}; \mathbb{Z})$ in each fiber $\mathbb{C}P^{n-1}$ for $i = 0, \dots, n-1$.

Recall from p 8 that:
$$\begin{array}{c} E \cong f^*(E_n) \\ \downarrow p \\ X \end{array} \iff g : E \to \mathbb{C}^\infty \text{ that is a linear injection on each fiber}$$

Let's projectivize g to get: $\mathbb{P}(g) : \mathbb{P}(E) \to \mathbb{C}P^\infty$

Let α be a generator of $H^2(\mathbb{C}P^\infty, \mathbb{Z})$,

then let $x = \mathbb{P}(g)^*(\alpha) \in H^2(\mathbb{P}(E); \mathbb{Z})$

Now, the powers x^i, $i = 0, \dots, n-1$ are the desired classes x_i since

a linear injection $\mathbb{C}^n \hookrightarrow \mathbb{C}^\infty$ induces an embedding $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^\infty$ for which

α pulls back to a generator of $H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$, so

α^i pulls back to a generator of $H^{2i}(\mathbb{C}P^{n-1}; \mathbb{Z})$

$$\begin{array}{ccccc} \text{generator of } H^{2i}(\mathbb{C}P^{n-1}; \mathbb{Z}) & \xleftarrow{\text{restrict on fiber}} & x_i & \xleftarrow{\mathbb{P}(g)^*} & \alpha^i \\ & & \cap & & \cap \\ & & H^{2i}(\mathbb{P}(E); \mathbb{Z}) & & H^*(\mathbb{C}P^\infty; \mathbb{Z}) \end{array} \Leftarrow \text{these are generators by p250 Hatcher}$$

(Note: any two linear injections $\mathbb{C}^n \hookrightarrow \mathbb{C}^\infty$ are homotopic through linear injections, so the induced embeddings $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^\infty$ of different fibers of $\mathbb{P}(E)$ are homotopic.
Also, we saw that any two choices of g are homotopic through maps that are linear injections on fibers, so the classes x^i are independent of choice of g)

Now, the Leray–Hirsch theorem says that $H^*(\mathbb{P}(E); \mathbb{Z})$ is a free $H^*(B; \mathbb{Z})$-module with basis $1, x, \dots, x^{n-1}$. Thus, x^n can be expressed uniquely as a linear combination of these basis elements with coefficients in $H^*(B; \mathbb{Z})$. Thus, we have a unique relation:

$$x^n - c_1(E) x^{n-1} + \dots + (-1)^n c_n(E) \cdot 1 = 0$$

for certain classes $c_i(E) \in H^{2i}(B; \mathbb{Z})$.

(For completeness we define $c_i(E) = 0$ for $i > n$ and $c_0(E) = 1$)

$H^*(B; \mathbb{Z})$: $c_i(E) x^i$ means

By definition of the module structure on $H^*(\mathbb{P}(E); \mathbb{Z})$ over

$$\mathbb{P}(\pi)^*(c_i(E)) \cup x^i$$

(Note: we have $(-1)^k c_k(E)$ since we want for the canonical bundle $E \to \mathbb{C}P^\infty$ for the relation to be
$x(E) - c_1(E) \cdot 1 = 0 \Rightarrow x(E) = c_1(E) = \alpha$
But for Stiefel–Whitney, the relation is $x^n + w_1(E) x^{n-1} + \dots + w_n(E) \cdot 1 = 0 \Leftarrow$ doesn't this mean $w_1(E) = -\alpha$?) $\mathbb{Z}_2$ coefficients: $+1 = -1$, $+\alpha = -\alpha$

To prove a),
Look at the pullback $f^*(E)=E'$:

$$\begin{tikzcd} E' \arrow[r, "\tilde f"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

If $g: E \to \mathbb{C}^\infty$ is a linear injection on fibers, then so is $g\tilde f$:

$E' \xrightarrow{\tilde f} E \xrightarrow{g} \mathbb{R}^\infty$, so we have $\mathbb{P}(g\circ\tilde f): \mathbb{P}(E') \to \mathbb{CP}^\infty$ with

$\mathbb{P}(g\circ\tilde f)^*(\underset{\uparrow}{\alpha}) \in H^2(\mathbb{P}(E');\mathbb{Z})$

generator of $H^2(\mathbb{CP}^\infty;\mathbb{Z})$

and of course this map factors through $\mathbb{P}(\tilde f)^*(\alpha)$, so the canonical class $x = x(E)$ for $\mathbb{P}(E)$ goes to the canonical class $x(E')$ for $\mathbb{P}(E')$.

Thus, $\mathbb{P}(\tilde f)^*(x^n - c_1(E)x^{n-1} + \cdots + (-1)^n c_n(E)\cdot 1)$

$= \mathbb{P}(\tilde f)^*\left(\sum_i \mathbb{P}(\pi)^*(c_i(E)) \cup (-1)^i x(E)^{n-i}\right)$

$= \sum_i \mathbb{P}(\tilde f)^* \mathbb{P}(\pi)^*(c_i(E)) \cup (-1)^i \mathbb{P}(\tilde f)^*(x(E)^{n-i})$

$= \sum_i [\mathbb{P}(\pi)\mathbb{P}(\tilde f)]^*(c_i(E)) \cup (-1)^i x(E')^{n-i}$

$= \sum_i \mathbb{P}(\pi')^* \mathbb{P}(f)^* c_i(E) \cup (-1)^i x(E')^{n-i}$ by commutativity of the diagram

Thus the relation $x(E)^n - c_1(E)x(E)^{n-1} + \cdots + (-1)^n c_n(E)\cdot 1 = 0$ defining $c_i(E)$

pulls back to: $x(E')^n - f^*c_1(E)x(E')^{n-1} + \cdots + (-1)^n f^* c_1(E)\cdot 1 = 0$ defining $c_i(E')$.

By uniqueness of the relation, $c_i(E') = f^* c_i(E)$.

Note: the naturality/functoriality gives us a quick proof that all Chern classes of a trivial bundle are zero:
If $p: E \to X$ is a trivial vector bundle, then since its classifying map is null homotopic, we have a map $f: X \to \{pt\}$, where $\pi: \mathbb{C}^n \to \{pt\}$ is the trivial n-dim complex vector bundle over $\{pt\}$, and so $E \cong f^*\mathbb{C}^n$. Then $c_i(E) = c_i f^*\mathbb{C}^n = f^*(c_i\mathbb{C}^n)$

$= 0$ since all cohomology classes of $\mathbb{C}^n$ are trivial (dealing with reduced cohomology here, I assume)

For b) $E_1 \hookrightarrow E_1 \oplus E_2$ gives inclusions $\mathbb{P}(E_1) \hookrightarrow \mathbb{P}(E_1 \oplus E_2)$
$E_2 \hookrightarrow E_1 \oplus E_2$ $\qquad \mathbb{P}(E_2) \hookrightarrow \mathbb{P}(E_1 \oplus E_2)$, $\quad \mathbb{P}(E_1) \cap \mathbb{P}(E_2) \cap \emptyset$

Let open sets: $U_1 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_1)$ deformation retract onto $\mathbb{P}(E_2)$
$U_2 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_2)$ $\qquad \mathbb{P}(E_1)$

A map $g: E_1 \oplus E_2 \to \mathbb{C}^\infty$ which is a linear injection on fibers restricts to a linear injection on fibers for both E_1 and E_2, so the canonical class $x \in H^2(\mathbb{P}(E_1 \oplus E_2); \mathbb{Z})$ for $E_1 \oplus E_2$ restricts to the canonical classes for E_1 and E_2 ?not entirely clear

Suppose $\dim E_1 = m$, $\dim E_2 = n$,
consider classes $\omega_1 = \sum_j c_j(E_1) x^{n-j} \to = 0$ in $H^m(\mathbb{P}(E_1); \mathbb{Z})$
in $H^*(\mathbb{P}(E_1 \oplus E_2); \mathbb{Z})$
$\omega_2 = \sum_j c_j(E_2) x^{n-j} \to = 0$ in $H^n(\mathbb{P}(E_2); \mathbb{Z})$

with cup product: $\omega_1 \omega_2 = \sum_j \left[\sum_{r+s=j} c_r(E_1) c_s(E_2) \right] x^{m+n-j}$

Thus ω_1 pulls back to a class in $H^m(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_1); \mathbb{Z}) \cong H^m(\mathbb{P}(E_1 \oplus E_2), U_2; \mathbb{Z})$
ω_2 " " $\qquad H^n(\mathbb{P}(E_1 \oplus E_2), \mathbb{P}(E_2); \mathbb{Z}) \cong H^n(\mathbb{P}(E_1 \oplus E_2), U_1; \mathbb{Z})$

Thus by the following diagram (with $\mathbb{Z}$ coefficients), $\omega_1 \omega_2 = 0$:

$$\begin{tikzcd}
H^m(\mathbb{P}(E_1 \oplus E_2), U_2) \times H^n(\mathbb{P}(E_1 \oplus E_2), U_1) \arrow[r, "\cup"] \arrow[d] & H^{m+n}(\mathbb{P}(E_1 \oplus E_2), U_1 \cup U_2) = 0 \arrow[d] \\
H^m(\mathbb{P}(E_1 \oplus E_2)) \times H^n(\mathbb{P}(E_1 \oplus E_2)) \arrow[r, "\cup"] & H^{m+n}(\mathbb{P}(E_1 \oplus E_2))
\end{tikzcd}$$

?why is this zero?

ω_1, ω_2

?why is $\omega_1 \in H^m$ and not H^*?

Thus, $\omega_1 \omega_2 = \sum_j \left[\sum_{r+s=j} c_r(E_1) c_s(E_2) \right] x^{m+n-j} = 0$,
but this is the defining relation for the Chern classes of $E_1 \oplus E_2$,
so $c_j(E_1 \oplus E_2) = \sum_{r+s=j} c_r(E_1) c_s(E_2)$

Property c) hold by definition

For d), Recall the canonical line bundle:

$$E = \{(\ell, v) \in \mathbb{C}P^\infty \times \mathbb{C}^\infty \mid v \in \ell\}$$

Here, $P(\pi)$ is the identity.

The map $g: E \to \mathbb{C}^\infty$ that is a linear injection on fibers can be taken to be

$g(\ell, v) = v$. Thus, $P(g): P(E) \to \mathbb{C}P^\infty$ is also the identity,

$$(\ell, \ell) \mapsto \ell$$

so $x(E) = P(g)^*(\alpha) \in H^2(P(E); \mathbb{Z})$ where α is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$

$= \mathbb{1}^* \alpha$

$= \alpha$ is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

The defining relation $x(E) - c_1(E) \cdot 1 = 0$ then says that

$c_1(E) = x(E) = \alpha$ is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$

To prove the uniqueness of the classes c_i we use the splitting principle:

First, we motivate with this example.

Def For an n-dim vector bundle $E \to B$, the associated flag bundle $F(E) \to B$ has total space $F(E)$ the subspace of the product $\underbrace{P(E) \times \cdots \times P(E)}_{n \text{ times}}$ consisting of n-tuples of orthogonal lines in fibers of E.

The fiber of $F(E)$ is thus the flag manifold $F(\mathbb{R}^n)$ consisting of n-tuples of orthogonal lines through the origin in $\mathbb{R}^n$.

i.e. the ith copy of $P(E)$

Now, the vectors in the ith line form a line bundle $L_i \to F(E)$

Then the direct sum $L_1 \oplus \cdots \oplus L_n$ is the pullback of E with respect to the projection $F(E) \to B$:

$$\begin{tikzcd}
L_1 \oplus \cdots \oplus L_n \arrow[r] \arrow[d] & E \arrow[d] \\
F(E) \arrow[r, "p"] & B
\end{tikzcd}$$

Think: for $(p_1, \dots, p_n) \in P(E) \times \cdots \times P(E)$, when you go to E, you basically "undo" the projectivization

(This follows since a point in the pullback consists of an n-tuple of lines $\ell_1 \perp \cdots \perp \ell_n$ in a fiber of E together with a vector v in this fiber, and v can be expressed uniquely as a sum $v = v_1 + \cdots + v_n$, $v_i \in \ell_i$)

?

Splitting Principle
For each vector bundle $\pi: E \to B$ there is a space $F(E)$ and a map $p: F(E) \to B$
such that the pullback $p^*(E) \to F(E)$ splits as a direct sum of line bundles,
and $p^*: H^*(B;\mathbb{Z}) \to H^*(F(E);\mathbb{Z})$ is injective.

proof? see flag bundle

(p 172 ~~Bredon~~ Hatcher:
Any identity among Chern classes of bundles that is true for bundles that are direct sums
of line bundles is true in general)

Now let's complete the proof:

Property ⓐ determines $c_1(E)$ for the canonical line bundle $E \to \mathbb{C}P^\infty$

Property ⓒ then determines ~~all~~ c_i's for this line bundle

Since the canonical line bundle is the universal line bundle, property ⓐ determines the classes c_i for all line bundles

Property ⓑ extends this to ~~a~~ sums of line bundles

Finally, the splitting principle implies that the c_i's are determined for all ~~line~~ vector bundles

---

Let $f: (\mathbb{C}P^\infty)^n \to G_n$ be the classifying map for the n-fold Cartesian product $(E_1)^n$ of the canonical line bundle E_1, and let $c_i = c_i(E_n)$

universal bundle for n-dim vector spaces

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Then the composition

$$\mathbb{Z}[c_1, \dots, c_n] \to H^*(G_n; \mathbb{Z}) \xrightarrow{f^*} H^*((\mathbb{C}P^\infty)^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha_1, \dots, \alpha_n]$$

send c_i to σ_i, the ith elem. symmetric polynomial.

In other words, the classes $c_i(E_n)$ generate a polynomial subalgebra $\mathbb{Z}[c_1, \dots, c_n] \subset H^*(G_n, \mathbb{Z})$

Thm. $H^*(G_n; \mathbb{Z})$ is the polynomial ring $\mathbb{Z}[c_1, \dots, c_n]$ on the Chern classes $c_i = c_i(E_n)$ of the universal bundle $E_n \to G_n$

Let $E \to B$
$\quad \parallel \quad \parallel$
$(E_1)^n \quad (G_1)^n$ $\to$ ~~[illegible]~~
$\to$ the n-fold Cartesian product $(E_1)^n \to (G_1)^n$ of the canonical line bundle over $G_1 = \mathbb{C}P^\infty$ with itself

This vector bundle is the direct sum

$\pi_1^*(E_1) \oplus \cdots \oplus \pi_n^*(E_1)$ where $\pi_i : (G_1)^n \to G_1$ is projection onto the ith factor

so $c((E_1)^n) = \prod_i (1+\alpha_i) \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n] \subseteq H^*((\mathbb{C}P^\infty)^n; \mathbb{Z})$

Thus $c_i((E_1)^n)$ is the ith elementary symmetric polynomial.

This can be done in general using splitting principle:
Given n-dim bundle $E \to B$, we know the pullback to $F(E)$ splits as a sum $L_1 \oplus \cdots \oplus L_n \to F(E)$.
Letting $\alpha_i = c_1(L_i)$, we see that $c(E)$ pulls back to
$c(L_1 \oplus \cdots \oplus L_n) = (1+\alpha_1) \cdots (1+\alpha_n)$ by the Whitney sum formula
$= 1 + \sigma_1 + \cdots + \sigma_n$

$\Rightarrow$ Thus c_i pulls back to σ_i

Thus, we have embedded $H^*(B; \mathbb{Z})$ in a larger ring $H^*(F(E); \mathbb{Z})$ such that $c_i(E)$ becomes the ~~[illegible]~~ ith elementary symmetric polynomial.

Let E be a vector bundle on ~~a~~ variety X of dim n

Chern classes $c_i(E) \in A_{n-i}(X)$

(recall first Chern class $c_1(\mathcal{L}) = [\mathrm{Div}(\tau)] \in A_{n-1}(X)$
for a line bundle $\mathcal{L}$ on variety X, for any rational section τ of $\mathcal{L}$)

We can measure nontriviality with the degeneracy locus of a collection of sections:
the locus where the sections become linearly dependent.

E trivial $\Leftrightarrow$ has r everywhere independent global sections $\tau_0, \dots, \tau_{r-1}$.
Thus a first measure of nontriviality is the locus where r general sections
$\tau_0, \dots, \tau_{r-1}$ are dependent

If we write $\tau : \mathcal{O}_X^r \to E$ for the map sending the ith basis vector to τ_i,
then this is the locus where τ fails to be a surjection, that is, where
the determinant of τ is zero. In other words, it is the zero scheme
of the section $\tau_0 \wedge \dots \wedge \tau_{r-1} \in \Lambda^r E$

~~$c_1(\Lambda^r E) := [\text{zero locus}] \in A_{\dim X - 1}(X)$~~

$c_1(E) := [\text{zero locus}] = c_1(\Lambda^r E) \in A_{\dim X - 1}(X)$

Thm There is a unique way of assigning to each vector bundle E on a smooth
quasi-projective variety X a class $c(E) = 1 + c_1(E) + c_2(E) + \dots \in A(X)$ in such a way
that:

a) (Line Bundles) If $\mathcal{L}$ is a line bundle on X then the Chern class of $\mathcal{L}$ is
$1 + c_1(\mathcal{L})$ where $c_1(\mathcal{L}) \in A^1(X)$ is the class of the divisor of zeros minus
the divisor of poles of any rational section of $\mathcal{L}$.

b) (Bundles with enough sections) If $\tau_0, \dots, \tau_{r-1}$ are global sections of $\mathcal{E}$, and the degeneracy locus D
where they are dependent has codimension 1, then $c_1(E) = [D] \in A^1(X)$

c) (Whitney's Formula) If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is a short exact sequence of vector
bundles on X, then $c(\mathcal{F}) = c(\mathcal{E})\, c(\mathcal{G}) \in A(X)$

d) (Functoriality) If $\phi : Y \to X$ is a morphism of smooth varieties, then
$$\phi^*(c(E)) = c(\phi^*(E))$$

For a bundle E of rank r that is generated by global sections, from S.3b) we get that $c_i(E)$ is the degeneracy locus on $r-i+1$ general global sections.

Another way to say this:

Prop Let E be a vector bundle of rank r on the smooth, quasi projective variety X, and let $W \subseteq H^0(E)$ be an m-dim vector space of sections generating E. If $\phi : X \to G(m-r, W)$ denotes the associated morphism sending $p \in X$ to the kernel of the evaluation map $W \to E_p$, then the ith Chern class $c_i(E)$ is the pullback $c_i(E) = \phi^*(\sigma_i)$ of the Schubert class $\sigma_i \in A^i(G(m-r, W))$.

For general (possibly non-globally-generated bundles), we have the following thm:

Thm Let E be a vector bundle of rank r on a smooth variety X, and let $\pi : \mathbb{P}E \to X$ be the projectivized vector bundle. Let ζ be the 1st Chern class of the dual S_E^* of the tautological bundle S_E on $\mathbb{P}E$.

a) the flat pull back map $\pi^* : A(X) \to A(\mathbb{P}E)$ is injective

b) the element $\zeta \in A(\mathbb{P}E)$ satisfies a unique monic polynomial $f(\zeta)$ of degree r with coefficients in $\pi^*(A(X))$

Def Let E be a vector bundle of rank r on a smooth variety X. The Chern classes $c_i(E)$ are the unique elements of $A(X)$ such that:

$$f(\zeta) = \zeta^r + \pi^* c_1(E) \zeta^{r-1} + \dots + \pi^* c_r(E)$$

That is, $A(\mathbb{P}E) = A(X)[\zeta]/(f(\zeta))$

Recall from algebraic topology:

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$$
$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) = H^*(G_1(\mathbb{C}^\infty);\mathbb{Z}) \cong \mathbb{Z}[\alpha] \text{ where } |\alpha|=2$$
(α is generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$)

Properties of chern classes: $c_i(E) \in H^{2i}(B;\mathbb{Z})$
a) $c_i(f^*(E)) = f^*(c_i(E))$
b) $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$, $c = 1 + c_1 + c_2 + \cdots$

Let's look at the n-product of the tautological line bundle E_1: pullback diagram:

$$\begin{tikzcd} (E_1)^n \arrow[r] \arrow[d] & E_1(\mathbb{C}^\infty) \arrow[d] \\ (G_1)^n \arrow[r, "\pi_1 \oplus \cdots \oplus \pi_n"] & G_1^{(\mathbb{C}^\infty)} = \mathbb{C}P^\infty \end{tikzcd}$$

where $(E_1)^n = \pi_1^* E_1 \oplus \cdots \oplus \pi_n^* E_1$

Now let's compute the Chern classes:

We know $H^*((G_1)^n;\mathbb{Z}) = H^*((\mathbb{C}P^\infty)^n;\mathbb{Z})$

$= H^*(\mathbb{C}P^\infty;\mathbb{Z}) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty;\mathbb{Z})$ by Kunneth (since all $H^*(\mathbb{C}P^\infty;\mathbb{Z})$ are f.g. free $\mathbb{Z}$-modules $\mathbb{Z}[\alpha]$)

$= \mathbb{Z}[\alpha_1] \otimes \cdots \otimes \mathbb{Z}[\alpha_n]$

$= \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$

$= \mathbb{Z}[c_1^{E_{1,1}}, \ldots, c_1^{E_{1,n}}]$ since we defined the 1st chern class as a generator of $H^2(\underset{G_1(\mathbb{C}^\infty)}{\mathbb{C}P^\infty};\mathbb{Z})$

So for each line bundle $E_{1,i} \to (G_1)^n$, we have $c(E_{1,i}) = 1 + c_1^{E_{1,i}} + 0 + 0 + \cdots 0 + \cdots$
$= 1 + \alpha_i$
(c: total Chern class)

Now, recall property b): $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$

Here: $c((E_1)^n) = 1 + c_1((E_1)^n) + c_2((E_1)^n) + \cdots + c_n((E_1)^n)$

$= (1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)$

$= 1 + (\alpha_1 + \cdots + \alpha_n) + (\alpha_1\alpha_2 + \cdots + \alpha_{n-1}\alpha_n) + \cdots + \alpha_1 \cdots \alpha_n$

$= 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n$

(σ_i: elementary symmetric polynomials)

Thus, inside $H^*((G_1)^n;\mathbb{Z}) = \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ generated by the ~~first~~ Chern classes of the line bundles E_1, live the Chern classes $c_i((E_1)^n) = \sigma_i$ of the n-dim vector bundle $(E_1)^n$

We can more generally use flag manifolds to identify the Chern classes c_i with elementary symmetric polynomials:

Given $E \xrightarrow{\pi} B$ (n-dim vector bundle) we have the pullback bundle from last time:

$$\begin{array}{ccc} \bar{\pi}^* E \cong L_1 \oplus \cdots \oplus L_n & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ F(E) & \xrightarrow{\bar{\pi}} & B \\ \cap\vert & & \\ \mathbb{P}(E) \times \cdots \times \mathbb{P}(E) & & \end{array}$$

By naturality (property a)):

$$c(L_1 \oplus \cdots \oplus L_n) = c(\bar{\pi}^* E) = \bar{\pi}^*(c(E))$$

$\shortparallel$ by b)

$$c(L_1) \cdots c(L_n)$$

$\shortparallel$

$$(1 + c_1(L_1)) \cdots (1 + c_1(L_n))$$

$\shortparallel$

$$1 + \sigma_1 + \cdots + \sigma_n$$

so $\sigma_i = c_i(\bar{\pi}^* E)$
$= \bar{\pi}^*(c_i(E))$ by naturality

In other words, since $\bar{\pi}$ is surjective, $\bar{\pi}^*$ is injective, so we get an embedding of

$$H^*(B; \mathbb{Z}) \hookrightarrow H^*(F(E); \mathbb{Z})$$
$$c_i(E) \longmapsto \sigma_i(c_1(L_1), \dots, c_1(L_n))$$

Now let's look at universal bundles $E_n \to G_n$.

Take the classifying map $f: (G_1)^n \to G_n$ of $(E_1)^n \to (G_1)^n$:

$$\begin{tikzcd}
f^* E_n = (E_1)^n \arrow[r] \arrow[d] & E_n \arrow[d] \\
(G_1)^n = (\mathbb{C}P^\infty)^n \arrow[r, "f"] & G_n
\end{tikzcd}$$

We know $c_i((E_1)^n) = \sigma_i(\alpha_1, \dots, \alpha_n)$ where $H^*((G_1)^n; \mathbb{Z}) = \mathbb{Z}[\alpha_1, \dots, \alpha_n]$ from before.

$$\begin{aligned} & \parallel \\ & c_i(f^* E_n) \\ & \parallel \\ & f^*(c_i(E_n)) \end{aligned}$$

Thus, the classifying map f for $(E_1)^n$ gives us a pullback f^* that sends the Chern classes of E_n in $H^*(G_n; \mathbb{Z})$ to the elementary symmetric polynomials in $H^*((\mathbb{C}P^\infty); \mathbb{Z})$.

In fact, we will now show that $H^*(G_n; \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$ so in fact this map f^* gives us an embedding of $H^*(G_n; \mathbb{Z})$ into the symmetric polynomials of $H^*((\mathbb{C}P^\infty)^n; \mathbb{Z})$.

Thm $H^*(G_n(\mathbb{C}^\infty); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$ where $c_i = c_i(E_n(\mathbb{C}^\infty))$ for the universal bundle $E_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$

Proof Sketch

We have the classifying map $f: (\mathbb{C}P^\infty)^n = (G_1)^n \to G_n$ pulling E_n back to $(E_1)^n$ as before.

We just showed that f^* contains the symmetric polynomials

But f^* only contains the symmetric polynomials because permutations of factors $(\mathbb{C}P^\infty)^n \to (\mathbb{C}P^\infty)^n$, permute the variables $\alpha_1, \dots, \alpha_n$ in $H^*(\mathbb{C}P^\infty; \mathbb{Z})$

So we've shown that f^* surjects $H^*(G_n)$ onto the symmetric polynomials in $H^*((G_1)^n)$.

We want to show that it is injective.

$\to$ Use CW decomposition of Grassmannians to show injection on cells

Flag variety / complete flag manifold

$Fl(E) = Fl(\mathbb{C}^m) = Fl(m) = Fl^{(m,m-1,\dots,1)}(\mathbb{C}^m)$

On $X = Fl(E)$ we have the universal / tautological filtration

$0 = U_0 \subseteq U_1 \subseteq U_2 \subset \cdots \subset U_{m-1} \subseteq U_m = E_X$

subbundles of E_X — trivial bundle on $X = Fl(E)$

Over a point in $Fl(E)$ corresponding to a flag $E_\cdot$, the fibers of these bundles are the vector spaces E_i of the flag.

We know from before that the Chern classes of the line bundles U_i/U_{i-1} are generators of the cohomology ring.

$L_i = U_i/U_{i-1}$, $x_i = -c_1(L_i)$

Prop3 p161 Fulton

Prop The cohomology ring of $X = Fl(m)$ is generated by the basic classes $x_1, \dots, x_m$ subject to the relations $e_i(x_1, \dots, x_m) = 0$, $1 \le i \le m$:

$H^*(X) = \mathbb{Z}[X_1, \dots, X_m]/(e_1(X_1, \dots, X_m), \dots, e_m(X_1, \dots, X_m))$

The classes $x_1^{i_1} \cdot x_2^{i_2} \cdots x_m^{i_m}$ with exponents $i_j \le m-j$ form a basis for $H^*(Fl(m))$ over $\mathbb{Z}$

Proof: V, $\dim V = r$, $\mathbb{P}(V)$ ← tautological line bundle $L \subseteq p^*(V)$

$V \downarrow Y$, $\mathbb{P}(V) \xrightarrow{p} Y$

Let $\zeta = -c_1(L)$.

Then $H^*(\mathbb{P}(V)) = H^*(Y)[\zeta] / \underbrace{(\zeta^r + c_1\zeta^{r-1} + \cdots + c_n)}$

Looks like Leray Hirsch: $c_1, \dots, c_m$ are generators of $H^*(F)$ mod the given relation

← fiber of $\mathbb{P}(V)$

(Fulton: classes $1, \zeta, \dots, \zeta^{r-1}$ form a basis of $H^*(\mathbb{P}(V))$ over $H^*(Y)$)

$H^*(Y) \otimes_{H^*(Y)} \mathbb{Z}[c_1, \dots, c_n, \zeta] / (\zeta^r + c_1\zeta^{r-1} + \cdots + c_n)$

$\cong H^*(Y)[\zeta]/(\zeta^r + c_1\zeta^{r-1} + \cdots + c_n)$

We construct $X = Fl(E)$ as a sequence of projective bundles:

$\mathbb{P}(E)$ has taut line bundle $U_1 \subseteq E \Rightarrow \text{rank}\, E/U_1 = m-1$,

Now: $\mathbb{P}(E/U_1) \hookleftarrow$ has taut line bundle U_2/U_1, $\text{rank}\, U_2 = 2$

$$\begin{array}{c} \mathbb{P}(E/U_1) \\ \downarrow \\ \cancel{\mathbb{P}(E)} \\ \vdots \\ \mathbb{P}(E/U_m) \end{array}$$

$H^*(Fl(m))$ side

Thus, $H^*(Fl(E)) \cong H^*(\mathbb{P}(E/U_1) \times \cdots \times \mathbb{P}(E/U_m))$

$$= H^*(\mathbb{P}(E/U_1)) \otimes \cdots \otimes H^*(\mathbb{P}(E/U_m))$$

$$\cong \{1, x_1, x_1^2, \ldots, x_1^{m-1}\} \otimes \cdots \otimes \{1, \cancel{x_m}\}$$

$$\left(\text{Thm: } H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})\right)$$

Since the x_i are the generators of the line bundle filtration of E_X, from before we saw that by the Whitney sum formula, the Chern classes of E_X are the elementary symmetric polynomials in the x_i. But since E_X is trivial, all chern classes vanish $\Rightarrow$ so all symmetric polynomials in the x_i are zero.

$R(m)$ side

Thus, we have a surjection:

$$\mathbb{Z}[X_1, \ldots, X_m] / (e_1(X_1, \ldots, X_m), \ldots, e_m(X_1, \ldots, X_m)) \to H^*(Fl(m))$$

$$X_i \mapsto x_i$$

To show it's an isomorphism, we want to show that

$$\text{so } X_1^{i_1} \cdots X_m^{i_m} \mapsto x_1^{i_1} \cdots x_m^{i_m}$$

$\uparrow$ basis for $H^*(Fl(E))$

WTS: $X_1^{i_1} \cdots X_m^{i_m}$ is a basis for $\mathbb{Z}[X_1, \ldots, X_m]/(e_1, \ldots, e_m)$ to prove injectivity $\Rightarrow$ iso

$$\prod_{i=1}^{m}(1-x_i t) = 1 - e_1 t + e_2 t^2 - e_3 t^3 + \cdots + (-1)^m e_m t^m = 1 \quad \text{since } e_i = 0$$

$$\Rightarrow \prod_{i=p+1}^{m}(1-x_i t) = \frac{1}{\prod_{i=1}^{p}(1-x_i t)}$$

$$\prod_{i=1}^{p} \frac{1}{(1-x_i t)}$$

$$= (1 + x_1 t - (x_1 t)^2 + \cdots) \cdots (1 + x_p t - (x_p t)^2 + \cdots)$$

$$= 1 + (x_1 + \cdots + x_p) t + (x_1 x_2 + \cdots + x_{n-1} x_n - x_1^2 - \cdots - x_p^2) t^2 + \cdots$$

$$= \sum_{i \geq 0} h_i(x_1, \ldots, x_p) t^i$$

t has pow at most: $m-(p+1)$ so $h_i(x_1, \ldots, x_p) = 0$ for $i > m-p$

$\Downarrow$

so starting with $h_{m-p+1}(x_1, \ldots, x_p) = x_p^{m-p} + \cdots = 0$, $1 \leq p \leq m$

we can successively cut down each variable by one power to get the basis