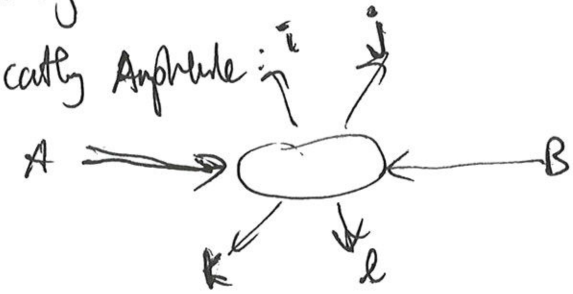
Intro: BCFW, Definitions, (Pos) Grass, Pos Geom, Example, locality & unitarity, Amplituhedra, the Amplituhedron, bi-adjoint ϕ^3 positivity

Quantum Field Theory:

Calculating Scattering Amplitude:



Lagrangian description of nature $\longmapsto$ experimental predictions of

Practical QFT class

Calculate amplitudes every way it could happen

$\rightarrow$ 10000's of terms

Jake Bourjaily slide

Final expression: Parke-Taylor Formula:

Gluon scattering $A[--++] = \dfrac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$

Why so simple?

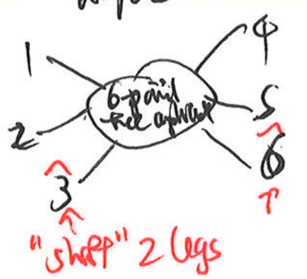
Better: Amplitudes

Obviously a lot of amplitudes literature, results & insights, but for this particular equation, an easy result uses BCFW recursion: (tree-level)

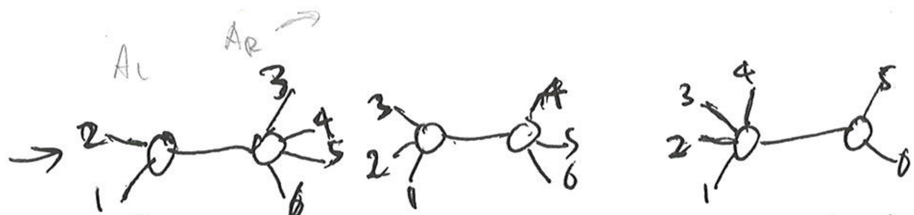
$$\boxed{M(0) = A_n = \sum_{\text{diagrams } I} \hat{A}_L(z^*)\,\frac{1}{P_I^2}\,\hat{A}_R^{\alpha}(z^*)}$$

(derivation I gave last semester)

Idea:



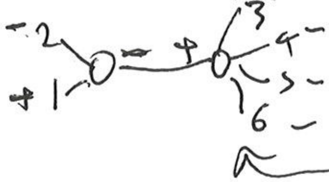
$\rightarrow$



① must always have at least 3 (otherwise no physical process, just another line)

But still a lot of diagrams (this is only tree level)

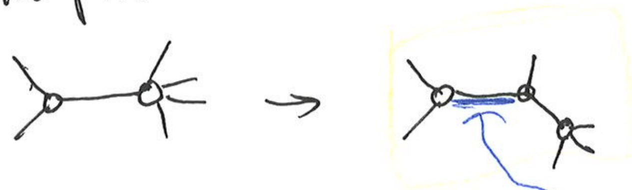
② Not quite: if you include helicity info: must have at least 2 −'s and 2 +'s unless a 3-vertex, which can be 2&1:



does not work

only one plus here

Next step [illegible] recursion:

 ... etc.

products of lower & lower-point amplitudes where internal lines are on-shell (imagine rather than taking all possible values of momenta for this virtual particle, it takes the value $p^2 = m^2$ ↑ mass)

but isn't it always on-shell at tree-level = classical?

BCFW very important

What is the Amplituhedron?

Intuitively, it is a geometric object which computes the (quantum field theory) scattering amplitude.

What is the mathematical definition of this geometric object?:

Let $Z \in \mathrm{Mat}^{>0}_{n,k+m}$.

The amplituhedron map $\tilde{Z} : Gr^{\geq 0}_{k,n} \to Gr_{k,k+m}$ is defined by $\tilde{Z}(C) := CZ$

- C: $k \times n$ matrix represents element of $Gr^{\geq 0}_{k,n}$
- CZ: $k \times (k+m)$ matrix representing element of $Gr_{k,k+m}$

Amplituhedron $A_{n,k,m}(Z) \subseteq Gr_{k,k+m}$ is the image $\tilde{Z}(Gr^{\geq 0}_{k,n})$

what is the exact relationship?

Note: this is the tree amplituhedron in planar limit (doesn't matter at tree level) for $N=4$ SYM theory

the Z's are momentum twistors, which are "dual" coordinates from spinor-helicity variables, so technically capturing the Wilson loop = "Wilsonhedron", but amplitude is the expectation value of a Wilson loop

Define Grassmannian

Review slides of BCFW

Q: open problem from Lauren & Lucas?

$\boxed{\text{Positive Grassmannian:}}$

$G(k,n)$ is space of all k-planes in $\mathbb{C}^n$

↗ 4th-dim as time

$G(2,4)$:

up to $GL(2)$ ↷ $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

↑ spans 2 plane in 4-dimensions

$Gr(2,5)$

$GL(2) \curvearrowright \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \end{bmatrix}$

so 6-dim $= k(n-k) = 2(5-2) = 2 \cdot 3$

← non-trivial quadratic Plücker relations

"Positive Grassmannian" = Totally nonnegative Grassmannian $G_+(k,n)$

~~Example~~: $\cancel{G_+(2,4)}$ Definition: All positive $k \times n$ matrices of rank k

all ordered maximal minors nonnegative

$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$ ← $\begin{bmatrix} a & b \\ e & f \end{bmatrix}$ $\begin{bmatrix} a & c \\ e & g \end{bmatrix}$, $\begin{bmatrix} a & d \\ e & h \end{bmatrix}$ $\begin{bmatrix} b & c \\ f & g \end{bmatrix}$ $\begin{bmatrix} b & d \\ f & h \end{bmatrix}$ $\begin{bmatrix} c & d \\ g & h \end{bmatrix}$ all ≥ 0

Example: $GL(1) \curvearrowright$ $G_+(1,3) = [c_1\ c_2\ c_3]$ so this is just projective space $\mathbb{P}^2$ c_1, c_2, c_3 all ≥ 0 and not all zero (or all ≤ 0)

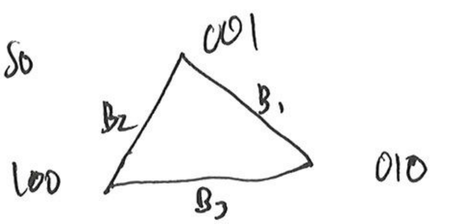
J pick a chart

$[1\ c_2\ c_3]$

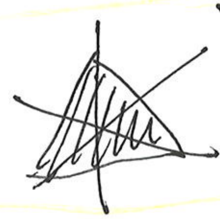
Boundaries: $B_2 = [1\ 0\ c_3]$ $c_3 \geq 0$, $[1\ c_2\ 0]$ $c_2 \geq 0$

(also the point at infinity, another chart: $[0\ 1\ d]$

so



(or think $\mathbb{RP}^2$



)

Can check:
Canonical differential form: $\Omega(G_+(1,3)) = d\log c_2 \wedge d\log c_3$ (without coordinate patch: $\frac{dc_1 dc_2}{c_1 c_2} + \frac{dc_1 dc_3}{c_1 c_2} + \frac{dc_1 dc_3}{c_1 c_3}$)

Intuition: Positive Grassmannians are generalizations of simplices

What is this $Gr^{\ge 0}_{k,n}$ thing?

First, notion of a <u>positive geometry</u>
= more general setting in which physicists think about $Gr^{\ge 0}_{k,n}$

Def. A D-dim positive geometry is a pair $(X, X_{\ge 0})$

X: irreducible complex projective variety of complex dimension D

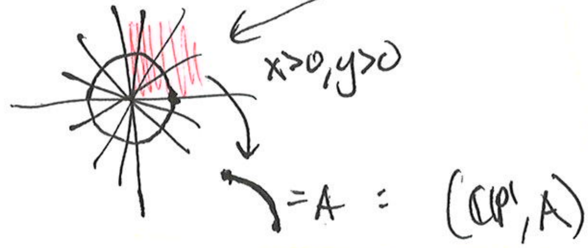
$\Rightarrow$ solution set in $\mathbb{CP}^N$ of a finite set of homogeneous polynomial equations (assume they have real coefficients)

$X(\mathbb{R})$ denotes the real part of X

$X_{\ge} \subseteq X(\mathbb{R})$ is a semialgebraic set: defined by finitely many real homogeneous equations <u>and</u> inequalities

Photo: restrict to $\mathbb{RP}^1$ inside $\mathbb{CP}^1$

$[x:y]$
$\downarrow$ real
$[x:y]$



+ a few technical assumptions
+ the following recursive axioms:

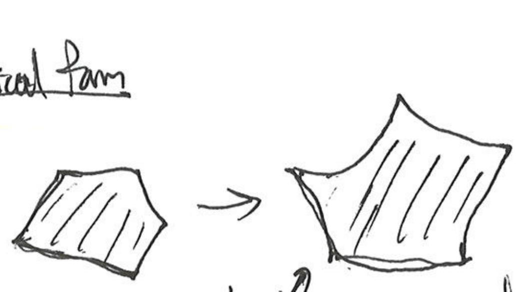
If $D=0$: X is a single point
$X_{\ge 0} = X$
The 0-form $\Omega(X, X_{\ge 0})$ on X defined to be ± 1 depending on the orientation of $X_{\ge 0}$

$D>0$: (1) Every boundary component $(C, C_{\ge 0})$ of $(X, X_{\ge 0})$ is a positive geometry of dimension $D-1$

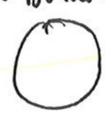
(2) There exists a unique nonzero rational D-form $\Omega(X, X_{\ge 0})$ on X constrained by the residue relation $\mathrm{Res}_C\, \Omega(X, X_{\ge 0}) = \Omega(C, C_{\ge 0})$ along every boundary component C, and no singularities elsewhere.

$\Omega(X, X_{\ge 0})$ called the <u>canonical form</u>

Intuition: nonlinear generalization of polytopes:

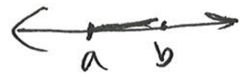


has boundaries of all codimensions.

Eg:  is <u>not</u> since no 0-dim boundaries

Example of positive Geometry.

Line segment $[a,b]$



If the ambient space does not admit any non zero holomorphic top forms, then the canonical form is unique for each positive geometry. So embed the space in projective space.

$X = \mathbb{C}$

$X_{\geq 0} = [a,b]$

~~if (x→∞): has pole at infinity~~ for projective invariance:

$$\Omega(X, X_{\geq 0}) = \frac{dx}{x-b} - \frac{dx}{x-a} = d\log\frac{x-b}{x-a} = \frac{(b-a)}{(x-b)(x-a)}dx$$

$$\frac{dx}{x-b} + \frac{dx}{x-a}$$

At $x=a$: substitute $z=x-a$, $z=0$

$$\Omega = \frac{dz}{z-b+a} - \frac{dz}{z}$$

$$\operatorname*{Res}_{z=0} = -1$$

At $x=b$: $+1$

$$\cancel{\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right]}$$

$$\cancel{\frac{\frac{1}{z-b}+\frac{1}{z-a}}{\frac{1}{z}-b} = \left(\frac{z}{1-bz}+\frac{z}{1-az}\right)\left(\frac{1}{z^2}\right) = \frac{1}{z}\left(\frac{1}{1-bz}+\frac{1}{1-az}\right) = \frac{1}{z-bz^2}+\frac{1}{z}}$$

$$d\left(\frac{1}{w}\right) = -\frac{1}{w^2}dw$$

$$\frac{dw}{\frac{1}{w}-b}$$

$$\frac{dw}{w(1-wb)} + \frac{dw}{w(1-aw)}$$

Projective simplices: $$\Omega(\mathbb{P}^m, \Delta_m) = \frac{\langle Z_1 Z_2 \cdots Z_{m+1}\rangle^m \langle Y\, d^m Y\rangle}{m!\,\langle Y Z_1 \cdots Z_m\rangle \cdots \langle Y Z_{m+1} \cdots Z_{m-1}\rangle}$$

Example for condition (2):

$\frac{dx\, dy}{xy(1-x-y)}$

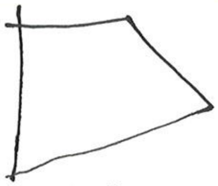
Res at $x=0$: $\frac{dy}{y(1-y)}$

Res at $y=1-x$

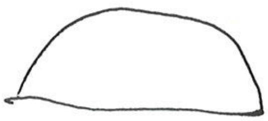
$z=1-x-y \Rightarrow y=1-x-z$

$dz=-dx-dy$

$\frac{dx(-dx-dz)}{x(1-x-z)z} = \frac{-dx\, dz}{x(1-x-z)z} \xrightarrow[z=0]{\text{Res}} -\frac{dx}{x(1-x-z)}$ ← residues at $x=0$ and $\underbrace{1-x-z}_{y}=0$



$\frac{dx\, dy\,(12-x-9y)}{x(2y-x)(3-x-y)(2-y)}$



$\frac{(3\sqrt{11}/5)\, dx\, dy}{(1-x^2-y^2)(y-(1/10))}$

The amplituhedron is (conjecturally) a positive geometry.
The positive Grassmannian $G_+(k,n)$ is a positive geometry.
This field of doing QFT via geometric objects could be categorized mathematically as "positive geometries."
(to pure mathematics)

Physics intuition:
BCFW:

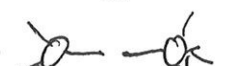
Think about the ~~integrand~~ integrand: $\int \frac{d^4 p_i}{p_i^2} \mapsto \frac{1}{p_i^2}$ ← for internal propagator

If BCFW puts this on-shell , then you are removing this ~~dof~~ ~~these~~ propagator so:

1. You have one less degree of freedom = going down in dimension 1:
   this is where the notion of "going to the boundary" (codim-1) comes from
2. Since this relation keeps dropping the dimension (dof) by 1 successively, you need boundaries of all codim
3. Removing the propagator is like taking a residue → hence the notion of "taking a residue of the differential form to go to the boundary"

Say

but for $\mathcal{L}_{n,k} = \int \frac{d^{n\times k}C}{GL(k)} \frac{1}{\prod M_i}\, \delta(\cdot\cdot)$
– this is a leading singularity / maximal cut so you have eliminated all degrees of freedom

Addendum: a) ~~unitarity~~ unitarity requires the right kind of factorization into  (lower point amplitudes → this is reflected in the boundary structure (although I won't talk much about it))
b) taking residues requires a choice of contour – which contour? It's picked out by the onshell diagram 

Back to the definition of the (tree) Amplituhedron $A_{n,k,m}(Z)$

$$k\left[\;\overset{n}{C}\;\right] \xrightarrow{\;n\left[\overset{k+m}{Z}\right]\;} k\left[\;\overset{k+m}{Y}\;\right]$$

$$Y_\alpha^I = C_{\alpha a} Z_a^I$$

$GL(k)$ — so $GL(k)$ — consider fixed

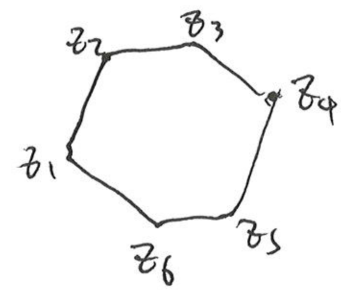
So $C_{\alpha a} \in GL(k,n) \Rightarrow Y_\alpha^I \in GL(k,k+m)$
but Y not necessarily positive

$$Z_a^I = \begin{array}{c|c|c} & m=4 & k=1 \text{ helicity sector} \\ & Z_I & \phi_1\cdot\chi_1 \cdots \phi_k\cdot\chi_k \\ & \text{momentum twistor} & \\ n \text{ particles} & \multicolumn{2}{c}{\left(\quad\quad\quad\quad\quad\quad\quad\right)} \end{array}$$

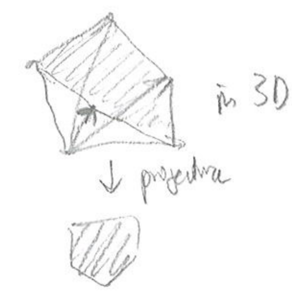
Simplest example: $k=1$, $m=2$

$$[c_1 \cdots c_n]\begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} = c_1 Z_1 + \cdots + c_n Z_n = Y$$

up to $GL(1)$, so projective: think about this as an n-gon:



Think:



The statement that Z-matrix is positive (all $\langle Z_{i_1} Z_{i_2} Z_{i_3} \rangle > 0$) is the statement that the polygon is convex. So taking positive linear combinations $c_1 Z_1 + \cdots + c_n Z_n$ makes sense since this stays inside the polygon.

So this is what positivity implies geometrically.
But what does it mean physically?

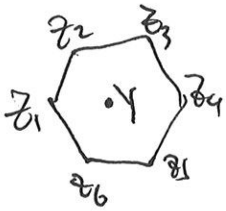
Locality & Unitarity

Locality: ~~poles of the integrand are simple poles~~ are "in the right places" correspond to physical particles going on-shell

Unitarity: as one of these internal particles goes on-shell, the amplitude factorizes correctly into a product of lower-point amplitudes

Bkg: Tree amplitudes are rational functions with simple poles located precisely where physical particles are exchanged

<u>Locality</u>:



Y inside the amplituhedron $\iff \langle Y Z_i Z_{i+1} \rangle > 0$ for all i

Z_i, Z_{i+1} describes a boundary

Generalize to m=4 ~~(details not shown)~~

$$\langle Y Z_a Z_b Z_c Z_d \rangle = \sum_{a_1,\dots,a_k} [C_{a_1} \cdots C_{a_k}][Z_{a_1} \cdots Z_{a_k} Z_a Z_b Z_c Z_d]$$

need to be put in order these ~~...~~ condition is positive

only way to not have a negative sign is if we have $Z_i Z_{i+1} Z_j Z_{j+1}$

so we have the boundaries:

$$\langle Y Z_i Z_{i+1} Z_j Z_{j+1} \rangle \to 0$$

but projecting through Y, this corresponds to $\langle Z_i Z_{i+1} Z_j Z_{j+1} \rangle = y_{ij}^2$ propagator

(or: $\langle Z_i Z_{i+1} Z_j Z_{j+1} \rangle \to 0$ means 2 lines in momentum twistor space intersect $\to$ corresponds to a null ray, that is, one of these particles becoming massless = going on-shell)

<u>Unitarity</u>:

Suppose one of these particles goes on-shell, we have a factorization channel:

so $\langle Y Z_i Z_{i+1} Z_j Z_{j+1} \rangle \to 0$, so WLOG, since the Y's are linearly independent (otherwise $\langle Y \cdot \rangle$ would be zero for everything)

we can say Y_1 in $Y = Y_1 \cdots Y_k$ is in the span of $Z_i, Z_{i+1}, Z_j, Z_{j+1}$

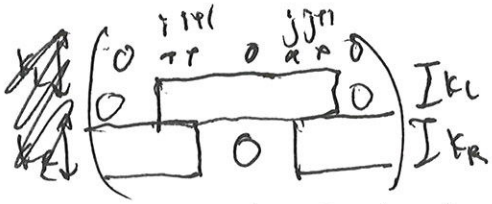
so if $k=2$ $\begin{pmatrix} 0 & \overset{i}{\times} & \overset{i+1}{\times} & 0 & \overset{j}{\times} & \overset{j+1}{\times} & 0 \\ & + & + & \mid & - & - & \end{pmatrix}$ by a $GL(k)$ transformation

~~If~~ this is nonzero, positive, fails by positivity, but then anything here fails $\Rightarrow 0$ (by positivity)

so the bottom row splits into 2 types: $\begin{pmatrix} 0 & \overset{i \; i+1 \quad j \; j+1}{\boxed{\quad\quad\quad\quad}} & 0 \\ \boxed{\quad\quad} & 0 & \boxed{\quad\quad} \end{pmatrix}$

Generalizes to higher-dimension:

specifying that row 1 is only nonzero in $i, i+1, j, j+1$, forces C to look like:



corresponding to a "factorized C":

Show that on the boundary, amplitudes splits into lower-dimensional amplituheda

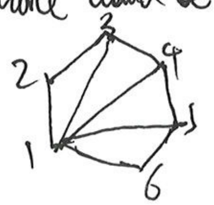
"Triangulations"

We saw for the case $A_{n,k,m}$ ($k=1$, $m=2$) that the final space was dim 2, but the original space was dim $n-1$.

More generally, dim $k(n-k) \longmapsto$ dim mk (and $n \geq m+k$)

"Triangulation" means subset of the domain that maps 1-1, so mk-dim subset of C that maps onto Y

In the case $k=1, m=2$ from before, a clearly nice choice would be cells $(1\; i\; i+1)$:



For $k=1, m=4$, this generalizes to the 4-dim cells $(1\; i\; i+1\; j\; j+1)$

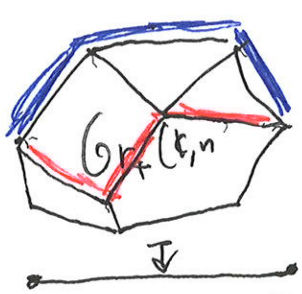
→ and these correspond exactly to one of the canonical BCFW decompositions of the $k=1$ (NMHV) tree amplitudes

For general k, As far as we know (and recently proven for $m=4$) ~~For higher k~~, BCFW always ~~(as far as we know)~~ corresponds to a ~~[illegible]~~ "triangulation" of the amplituhedron.

~~and for higher k (as far as we know) (and recently proved for m=4)~~ and ~~BCFW~~

and each ~~BCF~~W term in the triangulation comes from a positroid variety.

Here's the picture!



← positroids are the "boundaries"/stratification of $Gr_+(k,n)$

Different tilings = different BCFW recursions →

← amplituhedron

← canonical form is sum of pushforwards of canonical forms of positroids

Geometry/Math ⟷ Physics

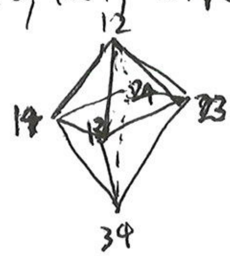
positroid tilings of the amplituhedron ⟷ BCFW

Lauren's Research: $m=2$ convex hull

Hypersimplex: polytope formed by Plücker vectors:

ex. $Gr(2,4)$: 1100, 1010, 1001, 0110, 0101, 0011

can be visualized in 3D as



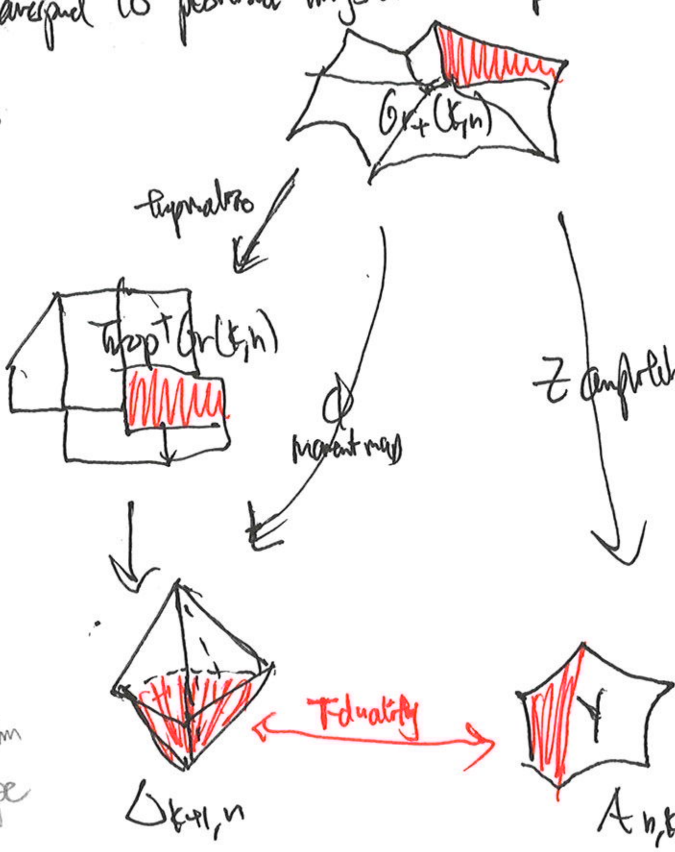
positroid subvarieties of $Gr(2,4)$ (or matroids) realizable (matroid polytope) e.g. just

are mapped into subpolytopes inside the hypersimplex via the moment map:

$$\mu(C) = \frac{\sum_{I\in\binom{[n]}{k}} |p_I(C)|^2 e_I}{\sum_{I\in\binom{[n]}{k}} |p_I(C)|^2}$$

What they found: using positive tropical Grassmannian, that positroid tilings of hypersimplex $\Delta_{k+1,n}$ correspond to positroid tilings of the amplituhedron $A_{n,k,2}(Z)$

Schematic



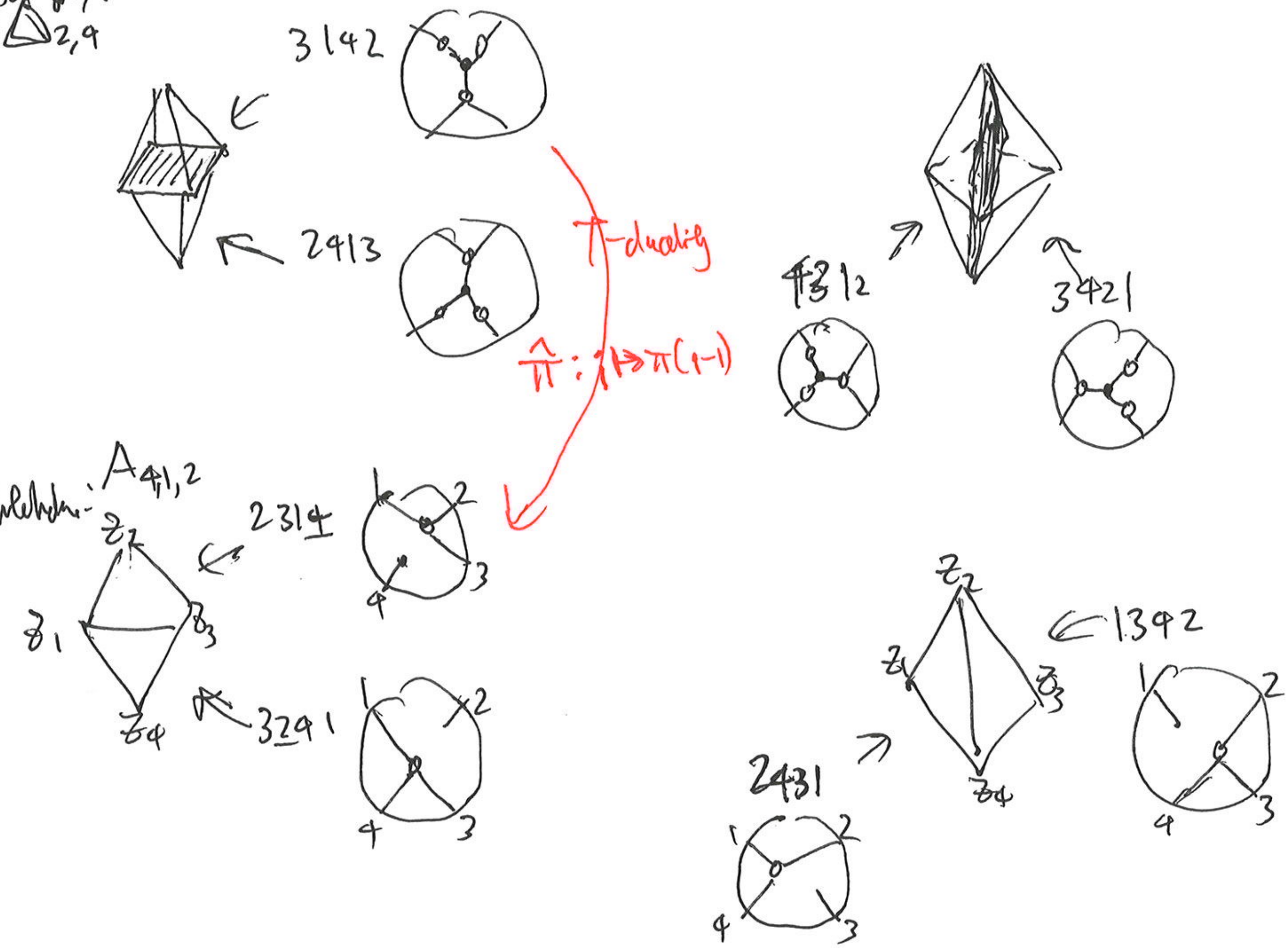
(n-1)-dim polytope

2k-dim non-polytopal subset of $Gr_{k,k+2}$

Example: $n=4$, $k=1$, $m=2$

Hypersimplex $\Delta_{2,4}$

Amplituhedron: $A_{4,1,2}$



How to Get the Amplitude from the Amplituhedron?

⇒ Via the Canonical Form

Any form on $Gr(k,k+4)$ looks like:

$$\Omega = \langle Y_1 \cdots Y_k\, d^4 Y_1\rangle \cdots \langle Y_1 \cdots Y_k\, d^4 Y_k\rangle \cdot \omega_{n,k}(Y;Z)$$

The amplitude is:

ignore

$$M_{n,k}(z_a,\eta_a) = \cancel{\int d^{N=4}\phi_1 \cdots d^4\phi_k \int \Omega_{n,k}(Y;Z)\,\delta^{4k}(Y;Y_0)}$$

$$= \int d^{N=4}\phi_1 \cdots d^{N=4}\phi_k\ \omega_{n,k}(Y_0; Z_a) \qquad Y_0 = \left(\frac{0_{4\times k}}{1_{k\times k}}\right)$$

(should be able to prove this by showing that it's true on every positroid cell, using the formula from last semester: $\int d^4\phi_1 \cdots d^4\phi_k \int \Omega^\Gamma \delta^{4k}(Y;Y_0) = \int \frac{d\alpha_1^\Gamma}{\alpha_1^\Gamma} \cdots \frac{d\alpha_{4k}^\Gamma}{\alpha_{4k}^\Gamma}\, \delta^{4k|4k}(C_{\alpha a}(\alpha) Z_a)$)

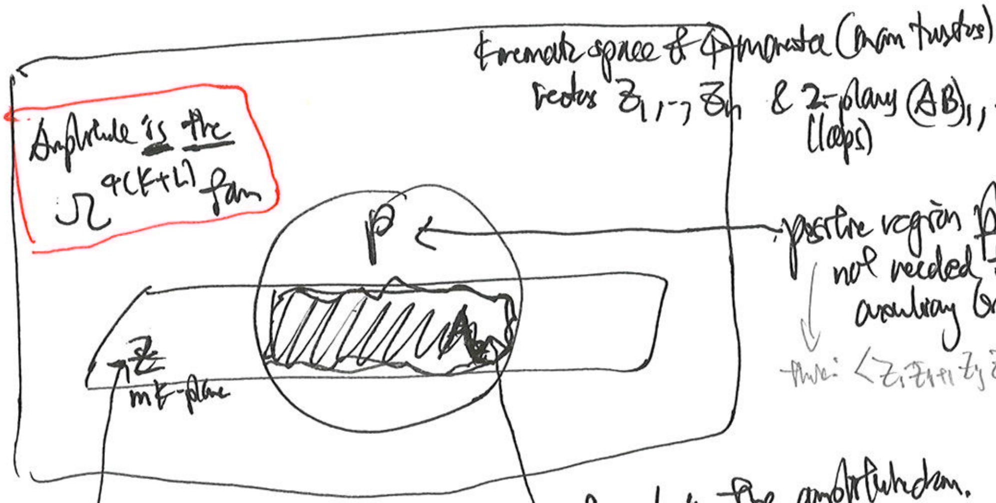
---

not super precise for rest of talk

But there's another way, relatively recent, using a new way of defining the amplituhedron directly in kinematic space

Here's the schematic (don't ask me about the details → not clear (to anyone?))

also involves loops



c.f. Unwinding the Amplituhedron in Binary

positive region plus a winding # condition (that was not needed for the original amplituhedron definition in auxiliary Grassmann space)

Think: $\langle Z_i Z_{i+1} Z_j Z_{j+1}\rangle \geq 0$, $\langle AB_\alpha\, i\, i{+}1\rangle \geq 0$, $\langle AB_\alpha AB_\beta\rangle \geq 0$

The intersection is the amplituhedron.

AND

The Amplitude $\Omega^{4(K+L)}$ is the unique $4(K+L)$-form that, pulled back to every choice of ~~m k~~-plane $\mathcal{Z}$ (no reference to positive region P) gives you the canonical form of $A_{\mathcal{Z}}$.

$\mathcal{Z}$ is any choice of $Z_1,\dots,Z_n$ shifted affinely by a fixed k-vector

In this formulation, they saw that the amplitude was exactly the differential form $\Omega\big|_{dZ\to X}$

i.e.: $A^{tree}_{n,k}(Z^A, X^A) \xrightarrow{X_i\to dZ_i} \Omega_{n,k}(Z^A, dZ^A)$

So the amplitude <u>is</u> a differential form!

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2 color indices otherwise $f^{abc}\phi^a\phi^b\phi^c = 0$
adjoint

Similar in the case of <u>bi-adjoint</u> ϕ^3 theory

Here, no supersymmetry, so no Grassmann variables, rather, the amplitude will be the coefficient $\rightarrow$ $(\dots)\, dX$ of the differential form

By analogy,

① A positive region: $\Delta_n: X_{i,j} \geq 0$, all $1 \leq i < j < n$

$$X_{i,j} = s_{i,i+1,\dots,j-1} = (s_i + s_{i+1} + \dots + s_{j-1})^2$$

② A (choice of) subspace(s): ~~simple exercise~~

$$s_{ij} = X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1} \quad \text{(simple exercise)}$$

condition: Let $X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1} = c_{ij}$ ~~constant~~ for <u>non-adjacent</u> i,j

$(= -s_{ij})$

Why? not sure: the idea is that this reduces the degrees of freedom for the non-independent variables to cut out a space of the right dimension, and it <u>works</u>

③ Differential form: cyclic scattering form $\Omega_n^{(n-3)}$ pulled back to the subspace is the canonical form of the associahedron.

Example: $n=4$

variables $(p_1+p_2)^2 = s$, $(p_1+p_3)^2 = u$, $(p_1+p_4)^2 = t$

$s+t+u = 0$ ($= \sum m_j^2$ but massless particles here)

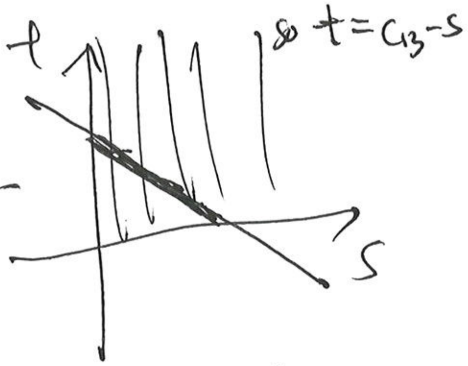
Positive region: $s, t \geq 0$

Subspace: $\underset{-u}{\overset{\|}{-s_{13}}} = c_{13} = -\underset{0}{\cancel{X_{14}}} - \underset{0}{\cancel{X_{23}}} + X_{13} + X_{24}$

$+ s_{12} + s_{23}$

$+ s + t$

so $t = c_{13} - s$



Planar scattering form: $\Omega = \frac{ds}{s} - \frac{dt}{t} = d\log\left(\frac{s}{t}\right)$

pull back to this line where $s+t=c$, $ds = -dt$

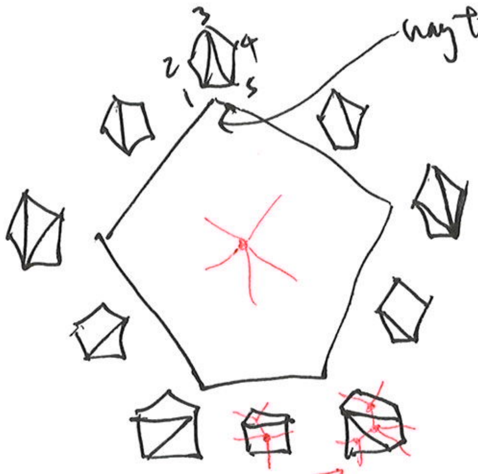
$\left(\frac{1}{s} + \frac{1}{t}\right) ds$

Amplitude!

$= A_4 \, ds$

A_4 ●——● via the associahedron

For A_5:



way to put 1234 in parentheses

Diagonals $\overline{13}$ $\overline{35}$

$\downarrow$ $\downarrow$

(12) (34)

This is supposed to be somewhat showing the shape of the amplituhedron for biadjoint ϕ^3

This: dual graph

Amplituhedron supposed to encode: locality & unitarity

What are these tree-level positive geometries / amplituhedra

in the form of a BCFW-like recursion:

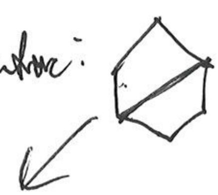
→ putting internal particles on-shell = going to the boundary

→ putting more & more on-shell = going to successively smaller boundaries

→ you have that structure exactly here:

and every diagram that can be reached by putting more propagators on-shell is on the boundary of this face

Furthermore, easy to see it exhibits the right factorization structure:



further triangulations are of the "lower-point amplitudes"