Positroids and Positroid Varieties

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Grassmannians

Let V be an n-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$)

For any integer $0 \leq k \leq n$, $Gr_k(V)$ is the set of all n-dimensional linear subspaces (through the origin) of V.

$Gr_k(V)$ can be given the structure of a smooth manifold of dimension $k(n-k)$

Examples

Examples:
(1) $Gr(0, n) = \{0\}$ the origin
(2) $Gr(n, n) = \mathbb{C}^n$ the whole space (is a point in the Grassmannian)

We have a natural duality: $Gr(k, n) \cong Gr(n-k, n)$ by considering the "orthogonal complement" $L^\perp$ of $L \in Gr(k, n)$. If $L = (E_k|C)$, then $L^\perp$ is the subspace spanned by the rows of $(-C^T|E_{n-k})$.

(3) $Gr(1, n) \cong \mathbb{P}^{n-1}$
(4) Thus by duality, $Gr(n-1, n) \cong \mathbb{P}^{n-1}$

Writing Grassmannians

We tend to write $Gr(k,n)$ as $GL(k)\backslash$[full rank $k \times n$ matrices], and think about it as k vectors spanning a plane up to invertible basis transformation inside the plane

Each point in $Gr(k,n)$ given by $GL(k)\backslash[k \times n]$, since it is full rank, has a set of columns $\lambda \in \binom{[n]}{k}$ whose determinant is nonzero, so we can use $GL(k)$ to set λ to be the identity so $(k \times n) \cong \begin{pmatrix} 1 & * & 0 & & 0 \\ 0 & * & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & 0 & & 1 \end{pmatrix}$ where the columns with all 0's except a single 1 are in λ

Plucker Embedding

•For $L \in Gr(k,n)$, we take a basis $\{v_1, ..., v_k\}$.
•Then we compute $v_1 \wedge \cdots \wedge v_k \in \Lambda^k \mathbb{C}^n \cong \mathbb{C}^{\binom{n}{k}}$, and then we projectivize $\mathbb{C}^{\binom{n}{k}} \to \mathbb{C}^{\binom{n}{k}-1}$.
•Easier way to think about it: this is all $k \times k$ determinants of the plane $GL(k) \curvearrowright \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_k & - \end{pmatrix} = V$
•(since for $A \in GL(k)$, for a $k \times k$ block of V given by columns λ, $det(A \cdot V|_\lambda) = det(A) \cdot det(V|_\lambda)$ we see that the $GL(k)$ action just scales every coordinate in the image by $det(A)$, so the image is well-defined up to nonzero scalar multiple: hence lives inside projective space.)

Plucker

The Plucker embedding defines a projective variety: thus it has an ideal defining the vanishing set. We might wonder about generators for this ideal - which is prime since the variety is irreducible. These generators are given by the Plucker relations, which are quadratic polynomials in the Plucker coordinates.

The Plucker relations are:

$$X_{i_1 \cdots i_k} X_{j_1 \cdots j_k} - \sum X_{i'_1 \cdots i'_k} X_{j'_1 \cdots j'_k} = 0$$

The sum is: take a fixed set of d of the $\{j_1, ..., j_k\}$ and sum over all d-element subsets of $\{i_1, ..., i_k\}$ where you switch them, e,.g. if we have $j = \{1, 2, 3\}$, $i = \{3, 4, 5\}$ and $d = 2$ with $\{2, 3\} \in j$, then we sum over: $\{2, 3\} \leftrightarrow \{3, 4\}$, $\{2, 3\} \leftrightarrow \{3, 5\}$, $\{2, 3\} \leftrightarrow \{4, 5\}$.

Matroids

Matroids were created to capture the combinatorial essence of linear dependence. There are many characterizations/equivalent definitions. Here's one:

A matroid $M \subseteq \binom{[n]}{k}$ satisfies the exchange axiom:
For all $I, J \in M$ and all $i \in I$ there exists a $j \in J$ such that $(I \backslash \{i\}) \cup \{j\} \in M$

Example:
$n = 3, k = 2$
$I = \{1, 2\}, J = \{1, 3\}$
$i = 2 : I \backslash i = \{1\}$, and let $j = 3 \in J$, then
$\{1\} \cup \{3\} = \{1, 3\} \in M$
$i = 1 : I \backslash i = \{2\}$, and let $j = 1 \in J$, then
$\{2\} \cup \{1\} = \{1, 2\} \in M$

Realizable Matroids

We are specifically interested in realizable matroids: matroids whose elements are all the sets of columns of a matrix that give a nonzero maximal minor.

We can take a matrix $A \in Gr(k, n)$, and we need to show that the columns of nonzero maximal minors do indeed satisfy the exchange relation given above.

Proof: $I, J \in M \Leftrightarrow \Delta_I(A) \neq 0$ and $\Delta_J(A) \neq 0 \Leftrightarrow \Delta_I \cdot \Delta_J \neq 0$. But we have $\Delta_I \cdot \Delta_J = \sum_{j \in J} \pm \Delta_{(J \setminus j) \cup i} \cdot \Delta_{(I \setminus i) \cup j}$ by the Plucker relations with one element i. Thus this sum must have at least one nonzero element which means there is a j such that $(I \setminus i) \cup j \in M$.

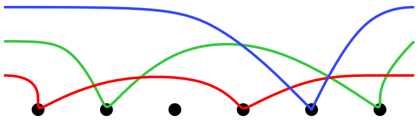
Positroids

What are positroids?

There are realizable matroids: matroids corresponding not to just any complex full rank $k \times n$ matrix, but rather corresponding to full rank $k \times n$ matrices with all maximal minors being nonnegative and real.

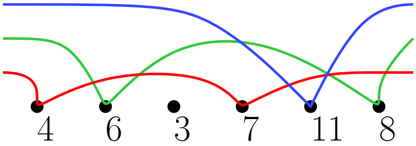
The amazing thing is that these matroids have very nice combinatorial properties.

Let's try to understand how this "mathematics of juggling stuff" works by doing an example. Look at the following picture of a juggling pattern that I created:



Each dot is a moment in time: the first dot is the time at the start, let's call it $t = 1$, the second dot is at $t = 2$
This juggling pattern repeats every 6 seconds.
There are 3 balls for each color;
We can think of the dots as the hand; note that this is a simplified model in which there is one hand.

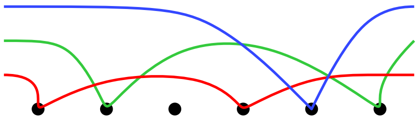
This sort of one-thing-at-one-time setup makes us think of mathematical permutations. And indeed, if we wanted get a list of numbers that described this pattern, we might get the numbers below:



We have written below each point the moment in time when the ball thrown at that moment lands. So the first point has a 4 because the (red) ball thrown at $t = 1$ lands at time $t = 4$. We can think of it in relation to our usual permutations in undergraduate abstract algebra by writing it like $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix}$. We call this a bounded affine permutation.

Wait, but this juggling pattern is periodic; there's no reason for why we chose the red ball with the 4 for the first moment in time - we could have started with the green ball with the 6 as $t = 1$ instead, just by shifting the whole thing over by one unit of time.

So what if instead of putting the time when each balls lands, let's put the units of time ahead of the present moment that the ball lands? Here's what we would get:



This second row of numbers is what we call the siteswap. Another way to see how we got this list of numbers is by taking the bottom row and subtracting the top row: $f(i) - i$ in the bounded affine permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix} \mapsto \begin{pmatrix} 4 & 6 & 3 & 7 & 11 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 \\ 3 & 4 & 0 & 3 & 6 & 2 \end{pmatrix}.$$

Now let's connect this to matrices and Grassmannians.

Here's the connection: a juggling pattern of length n with k balls corresponds to a set of k by n matrices.

Here, we can think of elements of Grassmannians as making the "symmetries" of a juggling pattern obvious:

(1) In particular, juggling patterns have a specific length of period (which is finite).

(2) Furthermore, juggling patterns are cyclic/periodic: they repeat for its fixed length interval.

(3) Juggling patterns have a fixed number of balls.

(4) Juggling patterns are permutations.

So what is the explicit map from juggling pattern to Grassmannian or vice-versa?

The map is easier in the opposite direction: from full rank k-by-n matrices to siteswaps. The map is that we take each column of a matrix, and assign the earliest number of the column after this given column for which this given column is in the span of the columns after it. For example:
$\begin{pmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$, which is a 4 by something-of-length-at-least-4 matrix, has first number 3. Why? Because the first column is not in the span of the 2nd and 3rd columns, but only in columns 2-4.

So now it should be clear how the map works. Clearly, there are a whole bunch of matrices that map to a given siteswap.

Even though it is more natural to take a matrix representative and calculate the earliest linear dependencies to get the siteswap, we can of course go in the other direction to get a matrix representative.Take for example the siteswap above: 340362. The following matrix does the job:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

There's another phrase that you might hear when people are talking about positroids, of ”cyclic rank conditions.” We only talked about the ”earliest place when a column is linearly dependent.” Where do the cyclic rank conditions come into play? It turns out that this siteswap actually gives information regarding the ranks of *any* consecutive number of columns. Let's describe how, and show how this is related to juggling.

First, let's switch back to doing bounded affine permutations.

Our bounded affine permutation was $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix}$.

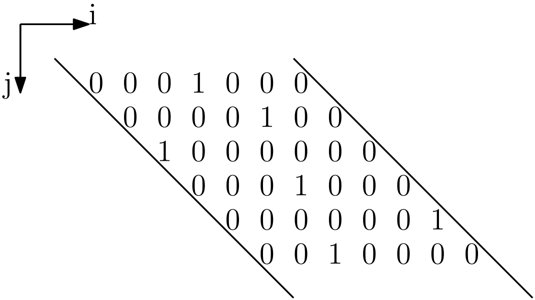
Let's stick our bounded affine permutation into a matrix. Let's put the first number in:

$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ Let's do the second column as well:

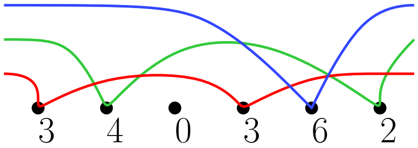
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Note that like a normal permutation matrix there is a single one in each row and column.

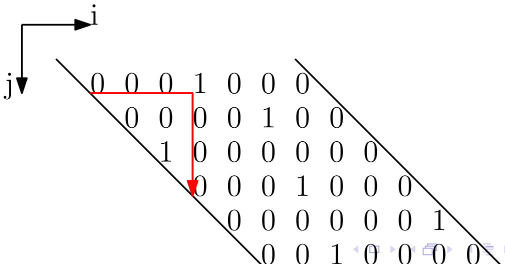
This is often times drawn with lines running from northwest to southeast showing the allowed region with horizontal axis i increasing to the right and vertical axis j increasing to the south:



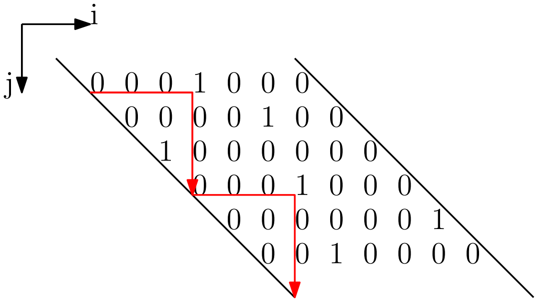
Let's recall our juggling pattern:



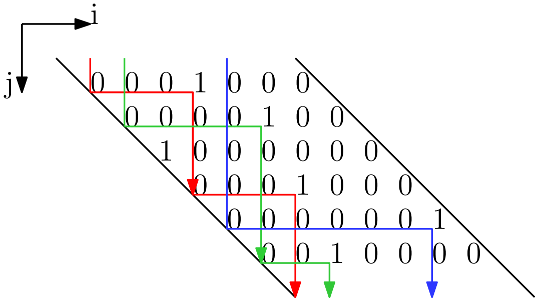
Let's draw the line on the permutation matrix corresponding to the first throw of the ball at time $t = 1$ that lands at time $t = 4$:



Let's draw the entire path of the ball in the juggling pattern represented by the red line:

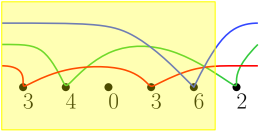


Let's show what the lines look like when we include all the balls in the juggling pattern:

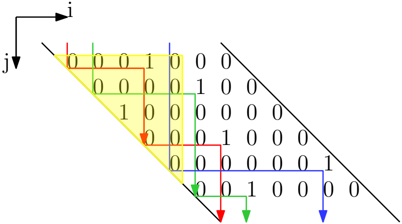


It's clear that what's going on is that we draw the lines horizontally from the left edge of the diagram until the line hits a 1, and then we go south until it hits the left edge of the diagram again, at which point we go horizontally right again.

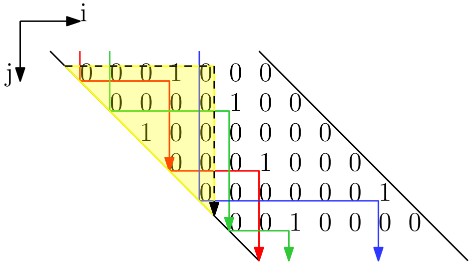
So finally, let's describe how we calculate the cyclic ranks. Let's do an example. Let's take the juggling pattern and let's look at $[i, j] = [1, 5]$:



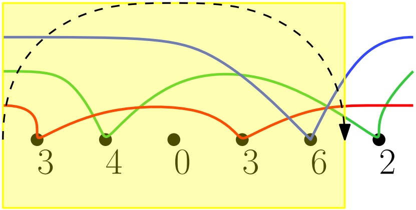
corresponding to the highlight in the permutation matrix:



There are 2 one's in the highlighted portion: at (1,4) and (3,3). The (1,4) corresponds to the dot at time $t = 4$ when the ball thrown at time $t = 1$ lands, and the $(3, 3)$ corresponds to the zero ball (no ball) at time 3. These are precisely the places of balls that are both thrown and landed within this interval $[1, 5]$. Perhaps we could make this clearer. Let the dotted black line in the permutation diagram be a hypothetical ball that would be thrown at time $t = 1$ and land at time $t = 5$:



This would correspond to a hypothetical ball of something like the dotted line below:



We have drawn it to start a little before time $t = 1$ and land a little after $t = 5$ to emphasize that it contains this interval (it includes both $t = 1$ and $t = 5$).

Here's the formula for rank:
$rank[i,j] = |[i,j]| - \{\#1's \text{ to SW}\} = (j - i + 1) - \{\#1's \text{ to SW}\}$. "SW" means "southwest." This should make sense from what we have just explained. In our example, $rank[1,5] = |[1,5]| - 2 = [(5-1)+1] - 2 = 3$. So the rank of this interval is 3.

This makes sense because given that the column when the ball was thrown is inside this yellow-highlighted region (and all columns up to and including when it lands are also inside the highlihgted region), it is dependent on the other columns - so it should not increase the rank of the matrix.

There's another way we could have seen this from the diagram: the rank is the number of balls that enter (or leave - they're the same since balls are not created or destroyed) the interval. In our yellow region of the juggling diagram, we see that all three balls - red, green, and blue - all enter the region and exit.

And then it becomes clear why the 1's inside the highlighted triangle do not add to the rank: because these are ball-lines representing throws that are both made and land within this time interval, so they do not add to the number of lines entering ot exiting the interval.

We've shown how to go from siteswap or bounded affine permutaton to this matrix of cyclic ranks. How do we go in the other direction?

Answer: From the diagram of ranks, whenever we spot this pattern: $\begin{pmatrix} r & r \\ r & r-1 \end{pmatrix}$, it means that this is where a ball drops.