Grassmannians

Lee Smooth Manifolds p22

Let V be an n-dim vector space (over $\mathbb{R}$ or $\mathbb{C}$)

For any integer ~~0≤~~ $k \leq n$, $Gr_k(V)$ is the set of all k-dim linear subspaces (through the origin) of V

Usually $V = \cancel{\mathbb{R}^n}$ or $\mathbb{C}^n$ so we denote $Gr_k(\mathbb{C}^n) = Gr(k, n)$

(Lee shows $Gr_k(V)$ can be given the structure of a smooth manifold of dimension $k(n-k)$)

But apparently lower dimensional subsets (e.g. in the positroid stratification) can have singularities

(Milnor-Stasheff show that it can be given the structure of a topological manifold via a quotient topology where we consider the stiefel manifold of all n-frames and map each n-frame to the point in the Grassmannian of the plane they span: then give the image the quotient topology (a subset is open iff its preimage is open)

Maybe: Mabelle's words → 00nn0n...
looks like the form of a BCFW bridge decomposition

Examples: $Gr(0,n) = \{0\}$ the origin
$Gr(n,n) = \mathbb{C}^n$ the whole space
We have a natural duality: $Gr(k,n) \cong Gr(n-k,n)$
by considering the "orthogonal complement" $L^\perp$ of $L \in Gr(k,n)$
(these show up when considering $\delta(C\cdot\tilde\lambda)\,\delta(C^\perp\cdot\lambda)$
as on p 29, 34, 70 of ABCGPT)
$L^\perp$ is the subspace spanned by the rows of $(-C^T \mid E_{n-k})$ for $L = (E_k \mid C)$

$Gr(1,n) \cong \mathbb{P}^{n-1}$
Thus, $Gr(n-1,n) \cong \mathbb{P}^{n-1}$

We tend to write $Gr(k,n)$ as $GL(k) \backslash \; k\begin{bmatrix} \overset{n}{\text{full}} \\ \text{rank} \end{bmatrix}$, and think about it as

. k vectors spanning a plane up to invertible basis transformations inside the plane

~~Then we can cover $Gr(k,n)$ by~~
Each point in $Gr(k,n)$ given by $GL(k) \backslash \begin{bmatrix} \overset{n}{abc \ldots} \end{bmatrix} k$, since it is full rank, has a set of columns λ whose determinant is non zero, so we can use $GL(k)$ to set λ to be the identity

so $\begin{bmatrix} \overset{n}{abc} \end{bmatrix} k \sim \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & \vdots \\ 0 & 0 & & 0 \\ 0 & 0 & & 1 \end{bmatrix}$
columns λ

This far for a point. Since every point has such a choice of λ, if we look at all $\binom{n}{k}$ choices of k-columns λ, every point in the Grassmannian must fall into one of these sets.
Thus, if we look at all sets $\begin{pmatrix} 1 \cdots 0 & \cdots & 0 \\ 0 \cdots 0 & \cdots & \vdots \\ 0 \cdots 0 & \cdots & 0 \end{pmatrix}$ where the other entries are free to vary, this (λ)

is a space isomorphic to $\mathbb{C}^{k(n-k)}$ (we have eliminated the $GL(k)$ action by using it to fix the k columns λ) so $Gr(k,n)$ ~~is covered by affine varieties~~ has an open cover by affine patches (= not just affine varieties but isomorphic to $\mathbb{A}^n$ itself), and since any 2 patches have nonempty intersection, $Gr(k,n)$ is an irreducible variety of dimension $k(n-k)$

don't include: don't know the proofs (see Cor 8.19 Gathmann) also don't you need to check the gluing of the sheaf of rings?

Plucker Embedding Irreducible projective variety

Since we have this mod $GL(k)$ on the left, we can't really think about $Gr(k,n)$ as existing inside $\mathbb{C}^{k\cdot n}$, ~~[illegible]~~ So how can we think about it as a geometric space?

Answer: via the Plücker embedding

For $L \in Gr(k,n)$, we take a basis $\{v_1, \ldots, v_k\}$

Then we compute $v_1 \wedge \ldots \wedge v_k \in \Lambda^k \mathbb{C}^n \cong \mathbb{C}^{\binom{n}{k}}$ and then we projectivize $\mathbb{C}^{\binom{n}{k}} \to \mathbb{CP}^{\binom{n}{k}-1}$

Easier way to think about: this is all $k \times k$ determinants of the

plane $GL(k) \curvearrowright \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = V$

(Since for $A \in GL(k)$, for a $k\times k$ block of V given by columns λ, $\det(A\cdot V|_\lambda) = \det A \cdot \det(V|_\lambda)$

we see that the $GL(k)$ action just scales ~~every~~ coordinate in the image by $\det A$,

So the image is well-defined up to nonzero scalar multiple: ~~hence~~ lives inside projective space

But the $v_1 \wedge \ldots \wedge v_k$ description is nice for proving that $G(k,n)$ is a projective variety, that is, that it is closed in $\mathbb{P}^{\binom{n}{k}-1}$ (defined by variety of ~~an~~ homogeneous ideal)

by using the lemma (8.14 Gathmann) that

For a fixed nonzero $w \in \Lambda^k \mathbb{C}^n$ with $k<n$ consider the $\mathbb{C}$-linear map

$$f: \mathbb{C}^n \to \Lambda^{k+1} \mathbb{C}^n$$
$$v \mapsto v \wedge w$$

Then $\operatorname{rank} f \geq n-k$, with equality holding iff $w = v_1 \wedge \ldots \wedge v_k$
i.e. w is a "pure tensor" (no sums)

The idea is that $\mathbb{C}^n = a_1 e_1 + \ldots + a_n e_n$

$$\text{so } f(\mathbb{C}^n) = (a_1 e_1 + \ldots + a_n e_n) \wedge \Big(\sum_{\substack{i \in I \\ |I| = k \\ I \subseteq \{1,\ldots,n\}}} b_I \, e_{i_1} \wedge \ldots e_{i_k} \Big)$$

This kills k of them, leaving $n-k$ left

So by the Plucker map, the image consists of pure tensors $v_1 \wedge \ldots \wedge v_k$, and then the lemma shows that this is equivalent to this element defining a map $\mathbb{C}^n \to \Lambda^{k+1}\mathbb{C}^n \cong \mathbb{C}^{\binom{n}{k+1}}$?

that has rank exactly $n-k$. Thus the image is exactly defined by the variety of all $(n-k+1)\times(n-k+1)$ determinants of this $\binom{n}{k+1} \times n$ matrix,

which are homogeneous polynomials in the coefficients in the $e_{i_1} \wedge \ldots \wedge e_{i_k}$ expansion of $v_1 \wedge \ldots v_k$ (which are exactly the coordinates of $\mathbb{P}^{\binom{n}{k}-1}$: the Plucker coordinates)

too long to explain

In particular, the (topological/smooth) manifold is compact

Now, we showed that the image of the Plucker map/embedding is a projective variety.

It is indeed an embedding:

take columns λ that are full rank and use $GL(K)$ to set these columns to be the identity:

$$\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & \vdots \\ \vdots & 0 & c & 0 \\ 0 & 0 & d & 1 \end{pmatrix}$$

then we can pick out each of these by choosing the λ columns that do not have this coordinate. Thus, computing all maximal minors, we get all the other entries on the nose, so the Plucker image picks out this matrix exactly - up to scalar because we could have used any other $GL(K)$, so we must projectivize.

However, although we have given elements of an ideal defining the variety set, we might wonder about generators for this ideal - which is prime since the variety is irreducible (see p2 or Gathmann 8.19)

These generators are given by the Plucker relations, which are quadratic polynomials in the Plucker coordinates. Given a matrix A, Plucker coordinates $(\Delta_{I_1}(A) : A_{I_2}(A) : \cdots)$ it's not

Surprising that there are relations among the determinants since potentially a very large number use the same columns.

The Plucker relations are:

$$X_{i_1 \cdots i_k} X_{j_1 \cdots j_k} - \sum X_{i_1' \cdots i_k'} X_{j_1' \cdots j_k'} = 0$$

The sum is: take a fixed set d of the $\{j_1, \dots, j_k\}$ and sum over all d-element subsets of $\{i_1, \dots, i_k\}$ where you swap them, e.g. if we have $j = \{1, 2, 3\}$, $i = \{3, 4, 5\}$ and $d = 2$ with $\{2, 3\} \in j$

then we sum over
$2 \leftrightarrow 3$ $3 \leftrightarrow 4$
$2 \leftrightarrow 3$ $3 \leftrightarrow 5$
$2 \leftrightarrow 4$ $3 \leftrightarrow 5$

where we keep the order and switching the order of two of them gives a minus sign

Grassmannian / Physics

Spinor-helicity:

$(p_0, p_1, p_2, p_3) \Rightarrow \begin{pmatrix} p_0 - p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 + p_3 \end{pmatrix}$

massless $\Rightarrow p^2 = 0 \Leftrightarrow$ determinant is zero

so decompose as outer product $\lambda^\alpha = \dfrac{z}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_0 - p_3 \\ -p_1 - ip_2 \end{pmatrix}$ $\leftrightarrow$ correspond to solutions of the Weyl equations

$$\tilde{\lambda}^{\dot\alpha} = \frac{z^{-1}}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_0 - p_3 & p_1 + ip_2 \end{pmatrix}$$

~~For these, invariant under SL(2) by Lorentz invariance, so we can move these around in the plane~~

Given a bunch of particles with momenta: $\lambda = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_n^1 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \end{pmatrix}$

$$\tilde{\lambda} = \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \cdots & \tilde{\lambda}_n^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \cdots & \tilde{\lambda}_n^2 \end{pmatrix}$$

~~Momentum conservation says~~ Invariant under SL(2) by Lorentz invariance, so we can move these (think of them as 2 n-dim vectors) in the plane they span. Furthermore momentum conservation says $\sum_j \lambda_j^{\alpha} \tilde{\lambda}_j^{\dot\alpha} = 0$: dot products are zero,

so they are in orthogonal planes

We can also define momentum twistor variables

$\mu_{\dot\alpha} := y_{\alpha\dot\alpha} \lambda^\alpha$

$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \to$ these are projectively invariant,

so modding out in projective space is torus action on the right

"close to configuration space"

similar to what we have here

~~M~~ Matroids

Matroids were created to capture the combinatorial essence of linear dependence.

There are many characterizations of matroids.

This is the main one we will be using

A matroid $M \subseteq \binom{[n]}{k}$ satisfies the exchange axiom:

For all $I, J \in M$ and all $i \in I$ there exists a $j \in J$ such that $(I \setminus \{i\}) \cup \{j\} \in M$

Example $n=3, k=2$

$\underset{I}{\{1,2\}}, \underset{J}{\{1,3\}}$

$i=2$: $I \setminus i = \{1\}$, and let $j=3 \in J$, then $\{1\} \cup \{3\} = \{1,3\} \in M$

$i=1$: $I \setminus i = \{2\}$, ~~and let~~ $j = 1 \in J$, then $\{2\} \cup \{1\} = \{1,2\} \in M$

We are specifically interested in realizable matroids: matroids which are the columns of a matrix ~~with nonzero determinant~~ giving a nonzero maximal minor.

or Grassmannian?

Since this is not changed by ~~basis~~ $GL(\mathbb{C})$ transformations, we can take a matrix $A \in Gr(k,n)$

And we should show that the columns of nonzero maximal minors do indeed satisfy the exchange relation of a matroid above:

Pf $I, J \in M \iff \Delta_I(A) \neq 0$ and $\Delta_J(A) \neq 0 \iff \Delta_I \cdot \Delta_J \neq 0$

But we have $\Delta_I \cdot \Delta_J = \sum_{j \in J} \pm \Delta_{(J \setminus j) \cup i} \cdot \Delta_{(I \setminus i) \cup j}$ by the Plücker relations with one element i

Thus this sum must have at least one nonzero element which ~~means~~ there is a j such that $(I \setminus i) \cup j \in M$.

(In fact, we even have $(J \setminus j) \cup i \in M$ as well; the strong exchange axiom)

? are exchange and strong exchange axioms the same?

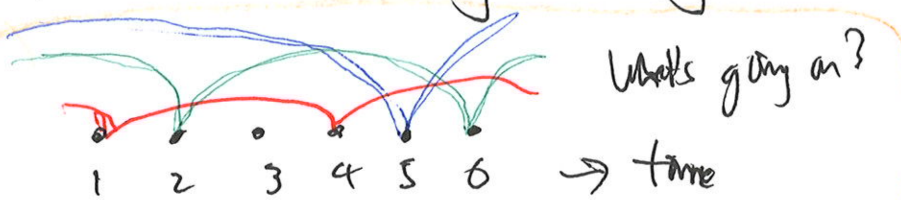
Positroids

Now let's look at positroids. What are positroids?

They are realizable matroids → matroids corresponding not to any complex ~~real~~ full rank $k \times n$ matrix – but rather correspond to full rank $k \times n$ matrices of all nonnegative real maximal minors. (If we have such a matrix, we can make it completely real by turning a set of full-rank columns into the identity matrix, which means all other entries must be real by the reality of all minors)

The amazing thing is that these matroids have very nice combinatorial properties –
~~they turn out to be given by~~

Let's start this discussion by introducing juggling patterns



3 colors → 3 balls

assume one hand – each moment at most a ball hits the hand ~~[illegible]~~ and is immediately thrown

This is an infinite periodic permutation. Our finite strip says

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix}$$

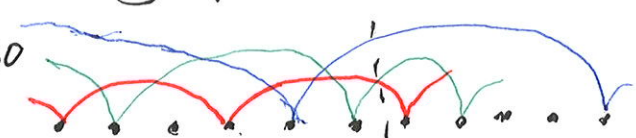
We call this a banded affine permutation.

Affine permutation

A bijection $f: \mathbb{Z} \to \mathbb{Z}$ satisfying the periodicity condition ~~[illegible]~~ $f(i+n) = f(i)+n \quad \forall i \in \mathbb{Z}$

Banded affine permutation: also satisfies $i \le f(i) \le i+n$

Siteswap notation: Since this is periodic, why must the starting time $t=1$ on the green ball instead? We can make it notation invariant by specifying the ~~number~~ time it takes for the ball thrown at that time to land, so



3 4 0 3 6 2

What's the connection to matrices / Grassmannians?

Take a full rank $k \times n$ matrix

$$\begin{array}{cc} & n \\ k & \begin{bmatrix} v_1 & \cdots \end{bmatrix} \end{array}$$

for each column, look at when $v_i \in \text{span}\{v_{i+1}, \ldots, v_{i+d}\}$,

then d is the number on the siteswap

Clearly, there are many matrices mapping to the same siteswap $\to$ thus we get a partitioning of the Grassmannian by positroid varieties that we will discuss later.

~~The dimensions make sense since the different colors/balls are the linearly dependent things, so should correspond to the number of rows k.~~

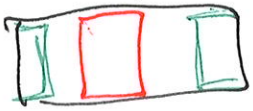
~~Intuitively~~ picking a starting time the first time we see a color drop, that means it wasn't in the span of any of the columns before, so it is linearly independent of anything before it $\Rightarrow$ it adds a rank. But any time afterwards that this ball lands, it is linearly dependent on the ones before. Hence, the number of colors/balls is the rank of the matrix.

This gives us the notion of a Grassmann necklace: (Def: sequence $(I_1, \ldots, I_n)$ of subsets $I_r \subseteq [n]$ such that
- if $i \in I_i$ then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for some $j \in [n]$
- if $i \notin I_i$ then $I_{i+1} = I_i$)

$I_1, \ldots, I_n$ all k-element subsets where I_i gives us the soonest columns after (and including) i when the new balls drop (when we get linearly independent elements)

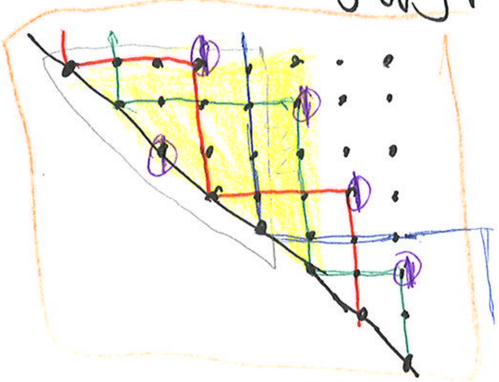
Easy exercise: show this is equivalent to the information given by siteswap

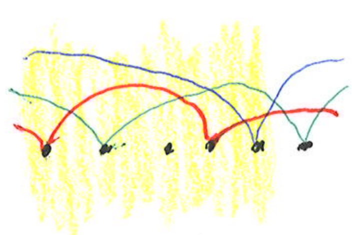
But in fact this siteswap, which told us when $v_i \in \text{span}\{v_{i+1}, \ldots, v_{i+d}\}$ actually gives us cyclic rank conditions, it gives us all ranks



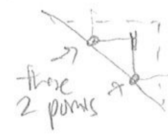
How? To see that, we will need to use a different diagram.

Let's draw out old juggling pattern on:





So think about a dot $\mathbb{1}$ as



it hit the interval, and
it relands again inside the interval

It's clear how this works (we can either think of it as a grid for the BAP or as jumping forward by the siteswap numbers)

But now let's put 1's at the corners.

Since this is an infinite permutation, there should be a single 1 in every row and column, zeros elsewhere

So let's say for $340362 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 7 & 11 & 8 \end{pmatrix}$ we want to know rank [1, 5], the first 5 column.

Let's highlight this portion in yellow

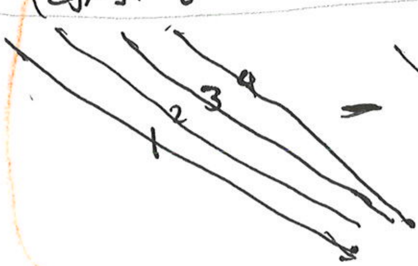
The rank of $[i,j]$ wants to be $j-i+1$ = the number of colums, here: $5 = 5-1+1$

But, each time a ball that already landed lands again, or we have a zero $\Rightarrow$ it does not increase the rank (it's linearly dependent on the others): this is exactly the number of 1's inside the shaded region.

Here it is $5-2=3$. So our matroid has rank $[1,5] = 3$

so if we did it all we would get:

$|[j,i]| = j-i+1$ − dots southwest



− Sum SW

= Ranks

$$\begin{matrix} [1,1] & [1,2] & [1,3] \\ & [2,2] & [2,3] \\ \text{rank} & & [3,3] \end{matrix} \quad \ldots$$

How do we go in the other direction? We take and reconstruct because we can pretty easily show that the ones occur when we have $\boxed{\begin{matrix} r & r \\ r-1 & r \end{matrix}} \leftrightarrow \boxed{1}$

means $[r[r-1]r]$ increases rank on either side, but stays the same when you add both

Finally, we've shown srtesurp/BAP $\iff$ Grassman necklace and we've hinted at the connection to positroids by showing correspondences to certain matrices, but here is the bijection explicitly:

Thm (Pos, Oh)
Let $\mathcal{I}=(I_1,\dots,I_n)$ be a Grassman necklace of type (k,n).
Then the collection: $$\mathcal{B}(\mathcal{I})=\{b\in\binom{[n]}{k} \mid b\geq_j I_j \text{ for all } j\in[n]\}$$
is the collection of bases of a rank k positroid $\mathcal{M}(\mathcal{I}):=([n],\mathcal{B}(\mathcal{I}))$.
Moreover, for any positroid P we have $\mathcal{M}(\mathcal{I}(P))=P$
(Thus, $\mathcal{M}$ and $\mathcal{I}$ are inverse bijections between the set of Grassman necklaces of type (k,n) and the set of rank k positroids on the set $[n]$)

First define:
The i-order on the set $[n]$ is the total order: $i<_i i+1<_i\dots<_i n<_i 1<_i\dots<_i i-1$
The Gale order on $\binom{[n]}{d}$ (with respect to $<_i$) is the partial order $\leq_i$ defined as follows:

for any two k-subsets $S=\{s_1<_i\dots<_i s_k\}\subseteq[n]$ and $T=\{t_1<_i\dots<_i t_k\}\subseteq[n]$ $\Rightarrow$ $S\leq_i T$ iff $s_j\leq_i t_j$ for all $j\in[k]$

For any rank k matroid $P=([n],\mathcal{B})$ let I_j be the lex min basis of P with respect to the order $<_j$ and denote $\mathcal{I}(P):=(I_1,\dots,I_n)$

Prop (Postnikov 16.3) For any matroid $P=([n],\mathcal{B})$ of rank k, the sequence $\mathcal{I}(P)$ is a Grassman necklace of type (k,n)

Example: Let's take $A=\begin{pmatrix}1&0&0&\overset{?}{-1}\\0&1&1&0\end{pmatrix}\in Gr(2,4)$

we know the positroid of nonzero maximal minors is $P=(\{1,2\},\underline{\{1,3\}},\{2,4\},\{3,4\})$

Lets do $\mathcal{I}(P)$ the lex min indep cols: $\mathcal{I}=(\underset{I_1}{\{1,2\}},\underset{I_2}{\{2,4\}},\underset{I_3}{\{3,4\}},\underset{I_4}{\{4,2\}})$ notice, different

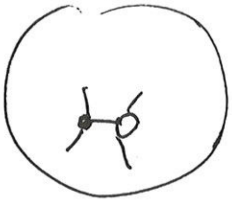
Let's calculate $\mathcal{B}(\mathcal{I}(P))$ — all $\binom{[n]}{k}$

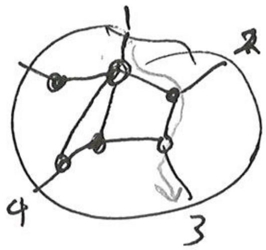
	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$I_1=\{1,2\}\leq_1$ $(1<2<3<4)$	V	V	V	V	V	V
$I_2=\{2,4\}\leq_2$ $(2<3<4<1)$	V	V	V	–	V	V
$I_3=\{3,4\}\leq_3$ $(3<4<1<2)$	V	V	V	V	V	V
$I_4=\{4,2\}\leq_4$ $(4<1<2<3)$	V	V	–	V	V	V

These are exactly the positroid

Lastly, plabic graphs: physics

on-shell diagrams for planar $\mathcal{N}=4$ SYM



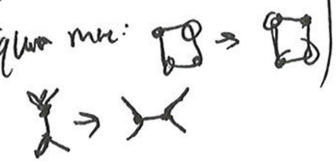


$1 \to 3$

left-right paths give us bounded affine permutations / siteswaps / decorated permutations

(up to certain moves

square move: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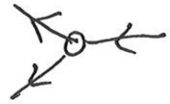
Another thing is perfect orientation



add variables:

consider

all paths from i to j

Then

is adding a path with λ times b = BCFW

$\lambda_a \to \lambda_a + z\lambda_b$

$\tilde\lambda_b \to \tilde\lambda_b - z\tilde\lambda_a$

Can use this BCFW to build up representative matrices:

$$\begin{array}{c} \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{pmatrix} \\ 7\ 8\ 9\ 4\ 5\ 6 \end{array} \xrightarrow[\alpha_1]{(36)} \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \alpha_1 \end{pmatrix} \to \cdots \to$$

matrix stuck into Grassmannian integral

$\lambda_6 \to \lambda_6 + \alpha_1 \lambda_3$

$$\oint_{C\in\Gamma_\sigma} \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))} \frac{\delta^{k\times 4}(C\cdot\tilde\eta)}{(1\cdots k)(n\cdots k-1)} \delta^{2k}(C\cdot\tilde\lambda)\, \delta^{2\cdot(n-k)}(\lambda\cdot C^{\perp})$$

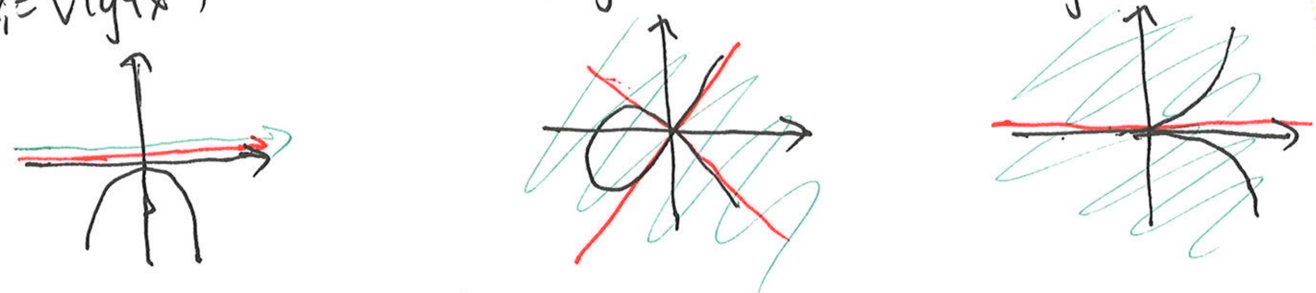
Our integral computes same sort of scattering amplitude

Smoothness

Example with cones

Consider the real points of

$X_1 = V(y+x^2)$ $\qquad$ $X_2 = V(y^2-x^2-x^3)$ $\qquad$ $X_3 = V(y^2-x^3)$



The tangent cone is in red.
The tangent space is in green.

Specifically

Def Tangent cone

Consider the blow up $\pi : \tilde{X} \to X$ of X at a. Its exceptional set $\pi^{-1}(a)$ is a projective variety (e.g. by choosing an affine open neighborhood $U \subseteq \mathbb{A}^n$ of $a = (a_1, \dots, a_n)$ in X and blowing up U at $x_1 - a_1, \dots, x_n - a_n$ (so the set $((x_1, \dots, x_n), [x_1-a_1 : \cdots : x_n - a_n])$) – the exceptional set is then contained in the projective space $\{a\} \times \mathbb{P}^{n-1} \subseteq U \times \mathbb{P}^{n-1}$)

The cone (e.g. putting the origin back in and considering the affine set) over this exceptional set $\pi^{-1}(a)$ is called the tangent cone C_aX of X at a. In the special case (of an affine patch) when $X \subseteq \mathbb{A}^n$ and $a \in X$ is the origin, we will also consider $C_aX \subseteq C(\mathbb{P}^{n-1}) = \mathbb{A}^n$ as a closed subvariety of the same ambient affine space as for X by blowing up at $x_1, \dots, x_n$

Ex.

For $y^2 - x^2 - x^3 = 0$, we have the blowup: $((x,y), [\tilde{x} : \tilde{y}]) \in \tilde{X}_2 \subseteq \tilde{\mathbb{A}}^2 \subseteq \mathbb{A}^2 \times \mathbb{P}^1$

and the blowup equation $\tilde{x}y - \tilde{y}x = 0 \iff$ away from the origin $(x,y) = \lambda(\tilde{x}, \tilde{y})$, $\lambda \in K^*$

Then $0 = \lambda^2(y^2 - x^2 - x^3) = \lambda^2 y^2 - \lambda^2 x^2 - \lambda^2 x^3$
$= \tilde{y}^2 - \tilde{x}^2 - x\tilde{x}^2 = 0$ on $\tilde{X}_2 \setminus \pi^{-1}(0)$ so also on its closure $\tilde{X}_2$.

(why?)

On $\pi^{-1}(0)$, i.e. if $x = y = 0$, this implies $\tilde{y}^2 - \tilde{x}^2 = 0$

$(\tilde{y} - \tilde{x})(\tilde{y} + \tilde{x}) = 0$ so the exceptional set is the points

$(1:1)$ and $(1:-1)$. So taking the cone, we get the solutions to the equation $(y-x)(y+x) = 0$ and this is the 2 lines $x = y$ and $\cancel{x} = -y$.

Tangent cone at the origin:

Computed by taking the homogeneous polynomial of all terms of minimal degree for all elements of the ideal (if it's a principal ideal then we only need to consider this for generators) and looking at their vanishing set

Now, what about the tangent space?

Let a be a point on a variety X. By choosing an affine neighborhood of a, we assume that $X \subseteq \mathbb{A}^n$ and that $a=0$ is the origin so that no polynomial $f \in I(X)$ has a constant term.

Then: $T_aX := V(\underbrace{f_1}_{\text{linear term}} : f \in I(X)) \subseteq \mathbb{A}^n$

So back to the example:

$C_0X_1 = V(y)$ $\quad C_0X_2 = V(y^2-x^2)$ $\quad C_0X_3 = V(y^2) = V(y)$

$T_0X_1 = V(y)$ $\quad T_0X_2 = V(0) = \mathbb{A}^2$ $\quad T_0X_3 = V(0) = \mathbb{A}^2$

Can define tangent space algebraically via:

$I(a)/I(a)^2 \cong \mathrm{Hom}_K(T_aX, K)$

so the dual of the tangent space is isomorphic to the vector space $I(a)/I(a)^2$ (over the field $\mathcal{O}_{X,a}/I(a) \cong K$). $\mathcal{O}_{X,a}$ the localization of $\mathcal{O}_X$ at a is a ~~regular~~ local Noetherian ring with maximal ideal $I(a)$ $\mathcal{O}_{X,a}$? so by commutative algebra $\mathrm{codim}_X\{a\} = \dim \mathcal{O}_{X,a} \leq \dim_K I(a)/I(a)^2$ and

$\dim_K I(a)/I(a)^2$ is the minimum number of generators for the ideal $I(a)$

Smooth / regular / nonsingular

→ When the maximal ideal has number of generators equal to the dimension of the ring, that is,

$\dim I(a) = \dim \mathcal{O}_{X,a}$

So for a variety X, a point $a \in X$ is smooth if its ring of local functions $A(X)_{I(a)}$ is regular

that is, if $\dim T_aX = \dim \mathrm{Hom}_K(T_aX, K) = \dim I(a)/I(a)^2 = \mathrm{codim}_X\{a\} = \dim X = \dim A(X)_{I(a)}$

localizing shouldn't change the dimension

In other words, qualitatively

The tangent cone has the right dimension but might not be linear

The tangent space is linear but might not be the right dimension

We always have $C_aX \subseteq T_aX$ since the linear terms are the lowest degree homogeneous terms

So if $\dim C_aX = \dim T_aX \iff C_aX = T_aX$

then X is smooth at a

Affine Jacobi Criterion

Let $a \in X$ be a point on an affine variety $X \subseteq \mathbb{A}^n$ and let $I(X) = \langle f_1, \dots, f_r \rangle$. Then

X is smooth at $a \iff$ the rank of the $r \times n$ Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} \text{ is at least } n - \operatorname{codim}_X \{a\}$$

(so it's equal to $n - \operatorname{codim}_X \{a\}$)

Proof

Let $x = (x_1, \dots, x_n)$ be the coordinates of $\mathbb{A}^n$ and $y := x - a$ be the shifted coordinates so a is the origin. By a formal Taylor expansion,

$$f_i(x) \cong f(a) + f'(a)y + \frac{f''(a)}{2!}y^2 + \cdots \text{ at } x \text{ near } a$$

so the linear term of f_i in these coordinates y is $\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(a) y_j$

By definition, the tangent space T_aX is zero locus of these linear terms: which is the kernel of the Jacobian matrix $J = \left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$

Thus, we know that

The point a is smooth on $X \iff \dim T_aX \overset{\text{or } \leq}{=} \operatorname{codim}_X\{a\}$

$\iff \dim \ker J \overset{\leq}{=} \operatorname{codim}_X\{a\}$

$\iff n - \operatorname{rank} J \overset{\leq}{=} \operatorname{codim}_X\{a\}$

$\operatorname{rank} J \geq n - \operatorname{codim}_X\{a\}$

Projective Jacobi Criterion

$X \subseteq \mathbb{P}^n$ a projective variety with homogeneous ideal $I(X) = (f_1, \dots, f_r)$, and $a \in X$.

X is smooth at $a \iff \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} \geq n - \operatorname{codim}_X\{a\}$

this is $r \times (n+1)$ matrix

But suppose we don't have generators of the ideal $I(X)$, but only a set $f_1, \dots, f_r$ that cuts out $\mathbb{A}^n \supseteq X = V(f_1, \dots, f_r)$ (so $(f_1, \dots, f_r)$ might not be radical). Then we still have one direction of the affine Jacobi criterion (or projective Jacobi criterion)

a) If $\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} \geq n - \operatorname{codim}_X\{a\}$ then X is smooth at a

b) If $\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j} = r$ (has max row rank) then X is smooth at a of local dimension $n-r$

Bad's Example

Jacobian criterion: $V(f_1,\dots,f_n)$ is smooth at $p \Leftrightarrow \mathrm{Jac}(f_1,\dots,f_n)(p)$ is full rank

$d_pf : T_p\mathbb{A}^d \to T_p\mathbb{A}^n$.

Note $T_pX = \ker d_pf$

Consider $X = V(xz-y^2) \subseteq \mathbb{A}^3$

$Y = V(xz-y^2) \subseteq \mathbb{P}^2$

X: $d_pf = \begin{bmatrix} z \\ -2y \\ x \end{bmatrix}$ which drops rank when $x=y=z=0$

Y: ① $U_1 = (z \neq 0) \cong \mathbb{A}^2$: $[\frac{x}{z} : \frac{y}{z} : 1]$

$Y \cap U_1 = V\left(\frac{x}{z}\cdot 1 - \left(\frac{y}{z}\right)^2\right) = \{\frac{x}{z} = \frac{y^2}{z^2}\}$ or just $\tilde{x} = \tilde{y}^2$ on this affine open

$d_pf = (1, 2y) \neq 0$. So Y is smooth on U_1

② U_2: $y \neq 0$

$Y \cap U_2 \Rightarrow xz - 1 = 0 \Rightarrow xz = 1$

$d_pf = [z, x]$ but $(0,0) \notin Y \cap U_2$ (can't have $0\cdot 0 = 1$)

③ U_3 is the same as U_1 by symmetry

Check: Using projective Jacobian criterion:

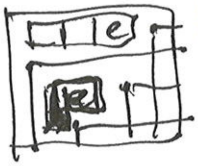
$d_pf = \begin{bmatrix} z \\ -2y \\ x \end{bmatrix}$ has rank at least $\underset{\text{since in } \mathbb{P}^2}{2} - \underset{Y \text{ has dim } 1}{\mathrm{codim}\{x\}} = 1$

Yes since $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is not in $\mathbb{P}^2$

Reason: $(X \setminus \{0\}) / K^\times = Y$

Rothe Diagram

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$



boxes are the inversion set:

$i < j$, $\pi(i) > \pi(j)$

the boxes are ~~[illegible]~~ $(i, \pi(j))$

(check:

(i,j)	$\longleftrightarrow$	$(\pi(i), \pi(j))$
$(1,2)$	$\longleftrightarrow$	$(4,1)$
$(1,3)$	$\longleftrightarrow$	$(4,3)$
$(1,4)$	$\longleftrightarrow$	$(4,2)$
$(3,4)$	$\longleftrightarrow$	$(3,2)$

Southeast corners are important b/c define essential rank conditions for a matrix schubert variety:

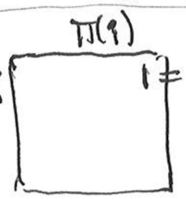
so Π_{4132} is all matrices A whose

$r_{i,j}$ rank from the northwest corner

is less than or equal to the rank of $4132 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(rank = ones to the northwest)

So think of this like



$\cdot = (1, \pi(1)) = (1,4)$

(important b/c later we'll see the BAP case is analogous)

being a box $\square$ means there is ~~nothing~~ no one to its left or top

so its dots are to the right and bottom

That is, if we denote $\underset{i\downarrow}{\overset{j}{\rightarrow}}$ for (i,j) to be a box, ① $\pi(i) > j$ and ② $\pi^{-1}(j) > i$

$\Updownarrow$

So if we relabel to be consistent with the notation above,

calling

then ① says $\pi(i) > \pi(j)$

② says $i < j$

Clearer:

means $\pi(i) > \pi(j)$

means this dot which is $(j, \pi(j))$ has $j > i$

So we should think of each permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix}$ and there are $4! = 24$ of them here as giving us a subset of the 4×4 matrices $\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ which is a 16-dimensional space

But there aren't ~~24~~ different pieces of $\mathbb{R}^{16}$ or $\mathbb{C}^{16}$ here (which would make us think of triangulations or BCFW), instead some are contained in others.

E.g. 1234 vs. 2134

$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

1234 puts no conditions since e.g. $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ says rank $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leq 2$ always true

$4321 = w_0$ $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$ is the smallest matrix Schubert variety / has the most restrictive conditions : $\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & \\ 0 & c & & \\ d & & & \end{pmatrix}$ $a, b, c, d \neq 0$

This is the most zeros you can get for ~~a~~ full-rank 4×4 matrix if you specify rank conditions starting from the north west

Thus 2134 has a single condition: rank $r_{11} = 0$ so this is the set: $\begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ where the other entries can be anything

Is it a stratification?

Another example: 3241 $\begin{pmatrix} & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \end{pmatrix}$ is satisfied by $\begin{pmatrix} 0 & 0 & 0 & \\ 0 & \cdot & & \\ 0 & & & \\ & & & \end{pmatrix}$

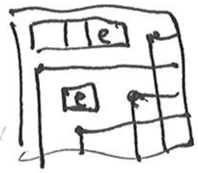
So we see why we cross out $\bullet\!\!\!-\!\!\!-$ right and below in this story/setup:

because a) $\bullet\!-\!-$ say we at (i,j) then this (i,j) submatrix has rank at most $\min(i,j)$

The dot at (i,j) because of the "$\lrcorner$" there means that our rank goes up, so $\leq$ rank adds no extra condition since $\operatorname{rank}(i,j) \geq \operatorname{rank}(i-1,j), \operatorname{rank}(i,j-1)$

If $j > i$ so (i,j) then clearly pushing the matrix out $\Rightarrow$ does not increase rank,

For this Rothe diagram of inversions, if we consider the southeast corners, we get the essential set.
If we have the ranks at this essential set, we can reconstruct the permutation:

Example 4132  look at ranks

$$\begin{array}{c|cccc} & 0 & 0 & 0 & \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{array}$$

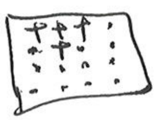
← should think of these as zeros: Then the essential set is the southeastern most instance of a 

So the algorithm is:  ← fill these in: either there's an essential box to the right or below with rank 0 (so ~~everything~~ everything to NW is also 0) OR the rank so it goes up to 1

Now go to  start filling in from corner:
Draw  a, b: then x is max{min{a,b}+1, 2}
unless there is an essential box with smaller number to the right or below

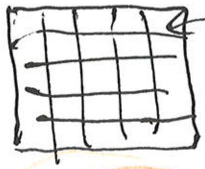
Positional case is analogous 

~~is the essential box of~~
The ~~essential~~ boxes are the northeastern-most boxes of rank  pieces,
so if there isn't something of this form to the northeast, then
you do:  x = max{min{a,b}+1, max for diagonal}

Perhaps even more important, if we consider the full Rothe diagram of inversions,
then if we push it to the left, we get the so-called
"bottom pipe dream": 
for the example above
to the right:
"top pipe dream": 

What's a pipe dream?

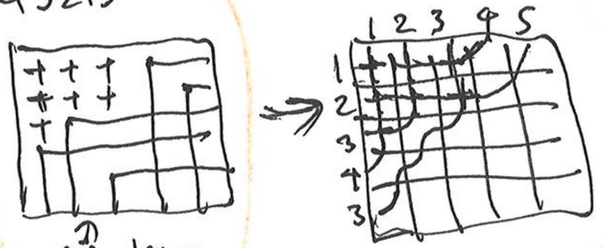
~~Think of as a tilted wiring diagram~~:



fill each square with + or)(

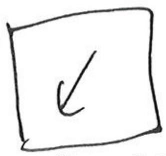
and follow the resulting pipes from SW to NE

Ex. 45213



Rothe diagram

going this way gets us the permutation

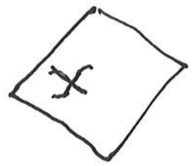


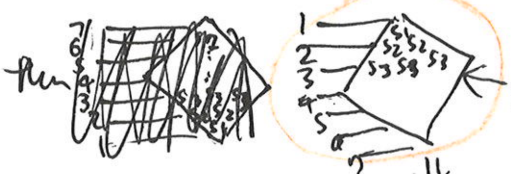
going this way gives us inverse permutation

(also sometimes people write 45213 as

so be careful)

We can think of this as a tilted wiring diagram





reading this way gives top pipe dream

square used: $s_3 s_2 s_1 s_4 s_3 s_2$

Bergeron–Billey:

For reduced pipe dreams (no double crossings), the ones for the same permutation can be related by chute and ladder moves:

chute

ladder

can move the cross and they'll end up at the same spot

Rothe / pipe 4

Deletion / Contraction

Why consider deletion/contraction?

Ziegler Polytopes p183:

Deletion and contraction of an "element" are fundamental operations in many areas... Deletion and contraction. There is a tremendous power in proofs "by deletion and contraction" which proceed by induction on the number of elements, and by putting a structure together from the information given by deletion and contractions of the same element.

We give the definition of deletion (aka "restriction") and contraction for matroids and show how it connects to other notions of deletion and contraction

<u>Restriction</u> of a matroid $M=(E,B)$ to $S\subseteq E$ is denoted $M|S$ with bases

$$B(M|S)=\{b\cap S \mid b\in B \text{ and } |b\cap S| \text{ is maximal among all } b\in B\}$$

Deletion for $\bar{S}\subseteq E$ is then restriction to $E\setminus \bar{S}$

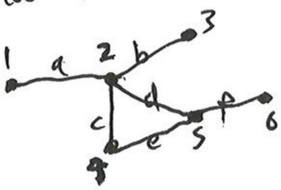
<u>Contraction</u> of this matroid by $T\subseteq E$ is denoted M/T with bases

$$B(M/T)=\{b-T \mid b\in B \text{ and } |b\cap T| \text{ is maximal among all } b\in B\}$$

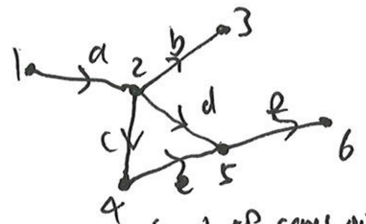
$\rightarrow$ They are actually dual operations. If $*$ is the dual operation of taking the complement of each of the bases, $(M/S)^* = M^*|E-S$

Let's see how this ties to usual contraction and deletion:

Take



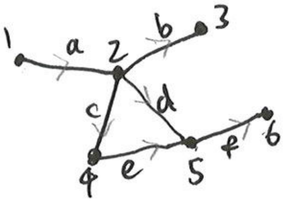
$\longrightarrow$ give the edges an orientation



$\downarrow$ -1 if comes out, 1 if goes in

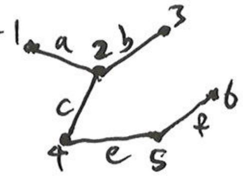
$$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline 1 & -1 & & & & & \\ 2 & 1 & -1 & -1 & -1 & & \\ 3 & & 1 & & & & \\ 4 & & & 1 & & -1 & \\ 5 & & & & 1 & 1 & -1 \\ 6 & & & & & & 1 \end{array}$$

$\leftarrow$ If we do positroid stuff, orientation doesn't matter since multiplying columns by scalars does not change rank conditions

So 

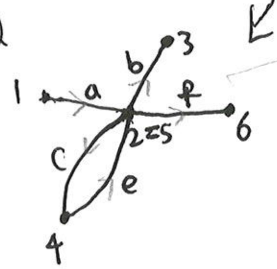
$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} -1 & & & & & \\ 1 & -1 & -1 & -1 & & \\ & 1 & & & & \\ & & 1 & & -1 & \\ & & & 1 & 1 & -1 \\ & & & & & 1 \end{pmatrix} \end{array}$$

← dimensions wrong

Delete d 

$$\begin{array}{c} d \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} -1 & & & & & \\ 1 & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

Contract d 

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} -1 & & & & & \\ 1 & -1 & -1 & -1 & & \\ & 1 & & & & \\ & & 1 & & -1 & \\ & & & 1 & 1 & -1 \\ & & & & & 1 \end{pmatrix} \end{array}$$

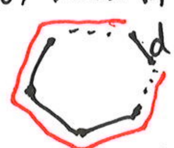
add these two columns

$$\begin{array}{c} \\ \\ 2=5 \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} -1 & & & 0 & & \\ 1 & -1 & -1 & 0 & 1 & -1 \\ & 1 & & & & \\ & & 1 & 0 & -1 & \\ & & & 0 & & 1 \\ & & & 0 & & \end{pmatrix} \end{array}$$

What happens to the matroids (nonzero maximal minors) of the matrix?

Deletion: every maximal minor that had d ~~now~~ no longer has nonzero determinant
This is restriction to the complement of d by the $|b \cap \{\text{edges} \setminus d\}|$ is maximal requirement

~~Contraction~~ Thus the matroids without d are the ones remaining after deletion

Contraction: There were matroids with d and without d. If without d, that means that d was in their span, which means in our setup, that this matroid contained a loop with d, except for d:  so once you contract d, you get a loop 

which is linearly dependent, so no longer a maximal minor.

Therefore, the matroids after contraction are precisely the matroids that had d → and now of course we eliminate d from these since d becomes $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

This is exactly the definition of contraction for matroids:

$$B(M/d) = \{ b - d \mid b \in B \text{ and } |b \cap d| \text{ is maximal among all } b \in B \}$$

this says $d \in b$

Proposition: positroids are closed under deletion, contraction, and duals.
(Proof: Ardila, Rincon, Williams Prop 3.5)

This deals with deletion/contraction combinatorially — for positroids (matroids).
What about positroid varieties?

For deletion, this is obvious: we want to get $\begin{array}{c} \\ \end{array}\begin{array}{c} i \\ \left(\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array}\right) \end{array}$ so this means

$i = 0$ in the siteswap.

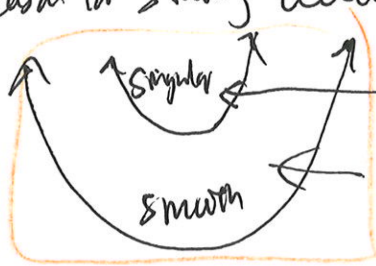
For contraction, from the defn of matroids, we want to get an i such that every matroid contains i and then get rid of it.

But every matroid contains i iff i is a linearly independent column, and so we can use $GL(k)$ to get $\begin{array}{c} i \\ \left(\begin{array}{ccccccc} & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right) \end{array}$, and then as usual we can get rid of it.

This says precisely that $i = n$.

Thus our goal is then to take an siteswap f of Π_f and get the largest positroid variety contained inside Π_f such that $f(i) = 0$ for deletion
$f(i) = n$ for contraction

Reason for studying deletion / contraction:



closed under going up

closed under going down

poset $f \leq g$ if f can be obtained from g by deletion / contraction

By Braterman-Brenda, if you consider bounded affine permutations for which positroid variety is singular, that's an order ideal.

So given the order ideal of singular positroid varieties, we wanted to describe the minimal elements and ask if it's a finite set.

(If you have an ideal, is the unique minimal generating set always finite?)

?

So what we were doing in the fall was trying to come up with something like pattern avoidance - e.g. a pattern containment criteria for positroid varieties.

Pattern avoidance: say 123 avoiding: means take a permutation matrix $\begin{bmatrix} 1 & & \\ & & 1 \\ 1 & & \end{bmatrix}$

cross out everything except 3

can't have something like $[1 \; 1]$ left

We looked at deletion and contraction on the permutation diagrams by looking at the maximal positroid variety with a 0 in column i

We saw that the permutation changes by

→ flip to get the red

So we have a kind of messy pattern avoidance sort of thing - of doing a bunch of these

We did notice that

① From the number of crossings in each row ~~gives~~ gives us a pipe dream of the ~~permutation~~ Grassmannian

② If you only allow weak Bruhat and BCFW type moves, the deletion involves just killing one of these rows (contraction killing columns)

Look into

→ but haven't gotten very far with this

Let's use this to look at smoothness for Schubert varieties, to make the positroid case easier, hopefully.

smooth are subvarieties

Whenever you have a group acting on a variety, ~~ok~~ then the singular locus will always be G-invariant. If you have a group of symmetries, those ~~symp~~ symmetries leave the singular locus alone but the notion of singularity does not require coordinates.

A point is singular if its Zariski tangent space ~~is~~ is too big, and the Zariski tangent space is independent of the coordinates used. Thus, if I have 2 points related by a group element, then I can use coordinates around one point or the other point, and then move them using the group - which gives us two different ways of getting coordinates at the same point. Then, they will both tell me if this point is singular or not. Then I pull it back and say therefore the other point is also singular or not. So because singularity is defined in terms of the Zariski tangent space, and it's independent of coordinates → that's why

Borel's ~~Fixed~~ Fixed Point Theorem

If a solvable group (such as a torus) acts on some nonempty proper variety, then the fixed point set is also nonempty.

$$\underset{\text{closed}}{(\Pi_f)_{\text{sing}}} \xrightarrow{T\text{-eqvt}} \underset{\text{proper}}{\Pi_f} \hookrightarrow Gr(k,n) \ni \lambda \text{ a } T\text{-fixed point}$$

Then

$$\text{So } (\Pi_f)_{\text{sing}} \neq \phi \underset{\text{Borel}}{\iff} (\Pi_f)_{\text{sing}}^T \neq \phi$$

That is, because the inclusion is T-equivariant, we know that T acts on $(\Pi_f)_{\text{sing}}$ and because it is closed inside Π_f which is proper, $(\Pi_f)_{\text{sing}}$ is also proper, and it's also nonempty, so by Borel, it has fixed points.

But we already know that the fixed points of the ~~torus~~ action are λ:

Thus, checking on the open affine cover U_λ,

$$\Pi_f \cap U_\lambda \text{ smooth} \iff \Pi_f \text{ smooth at } \lambda \text{ (coordinate fixed point)}$$

This is nice since this just means we need to check smoothness at $\binom{n}{k}$ points

(Note: ~~if~~ if we're trying to show this explicitly for the U_λ, we may have to use Bialynicki-Birula to show there is a ~~stream~~ circle in the torus pulling that open set to λ, so ~~if~~ if there were singularities in that open set, then we have a singularity all the way to λ, and by closedness of singularities, λ is singular as well.

Alright, so we've seen that we just need to check at the points $\lambda \in U_\lambda$.
What do we need to check?

We use a <u>theorem</u>

If $X \subseteq \mathbb{C}^n$ is a cone ($\mathbb{C}^\times$ dilation invariant), then

X smooth $\iff$ X smooth at 0 $\iff$ X linear

Note that since the singular locus is invariant under $\mathbb{C}^\times$ also (see the last page), if X is a cone, (because?) then the singular locus of X is also a cone

Note, we are working in $U_\lambda \cong \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{bmatrix} \cong \mathbb{C}^{k(n-k)}$ which is clearly dilation invariant, so is a cone.

The origin of this affine set is the point λ. So in case it isn't naturally obvious that the "problem point" of a cone is the origin, our results on the last page already showed

X smooth $\iff$ X smooth at 0 for $X = \Pi_f \cap U_\lambda$

And Π_f is a cone since scaling doesn't change rank conditions, so $\Pi_f \cap U_\lambda$ is a cone

So we just have to show the second $\iff$ that smoothness at the origin is equivalent to X being a linear space

For this we use the affine Jacobian criterion:

Since it's a cone, all equations defining the set are homogeneous polynomials of the same degree.

Say $X = V(f_1, \ldots, f_n)$

If we calculate $\left(\frac{\partial f_i}{\partial x_j}\right)(0)$ every term dies out except if f_i was linear. Therefore,

X must be defined by linear equations.

In our case, since we know that our subsets of Grassmannians are defined by determinantal conditions, we conclude that these determinantal conditions must be linear.

Let's do an example.

It's a fact that the smooth Schubert varieties are the ones isomorphic to sub Grassmannians.

See 1:55 Allen August 18 recording

$$\begin{tikzcd} (X_\lambda)_{sing} \arrow[r, hook] \arrow[loop below, "B_-"] & X_\lambda \arrow[loop below, "B_-"] \end{tikzcd}$$

By what we've said already B_- acts on the singular locus (it's B_- invariant)

Allen says each Schubert X_λ only has finitely many B_- orbits and each of their closures is a union of Schubert varieties, so

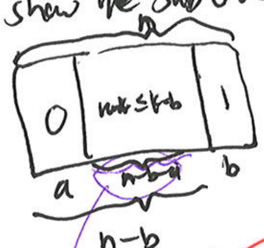
$$(X_\lambda)_{sing} = \bigcup_{\underset{\text{Bruhat order}}{\lambda < \mu}} X_\mu$$

Therefore, either $(X_\lambda)_{sing}$ is empty, or if it's not, it must contain the Schubert point.

In other words,

X_λ smooth (regular) $\iff$ X_λ smooth at $(0\ 1\ldots)$

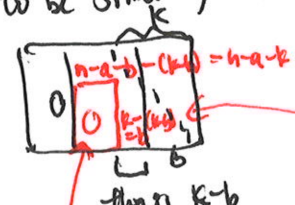
Let's show the sub Grassmannians inside Schuberts are smooth:



Gr(K, n-a), so V a k-plane inside $\mathbb{C}^{n-a}$

Inside here, since columns in b have full rank, we have Gr(K-b, n-a-b) so these are planes containing $\mathbb{C}^b$ inside $\mathbb{C}^{n-a}$: $\mathbb{C}^b \subseteq V \subseteq \mathbb{C}^{n-a}$

Now, to be smooth, it must be smooth at the schubert point: ? isn't the Schubert point $\boxed{0\ 1\ldots}$ Does he mean: contains the Schubert point?

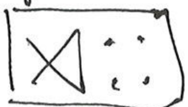


these must be zero, otherwise would add ranks such that are redundant.

Therefore, we are left with the subgrassmannian Gr(K-b, n-a-K)

And the conditions $x_{ij} = 0$ inside the red box are linear.

Example

 $\in Gr(2,4)$

$\det = 0 \iff \mathrm{rank}[1,2] \le 1$

$\lambda = \{3,4\}$: $\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}$: $ad - bc = 0$: singular

$\lambda = \{1,2\}$ $\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$: $1 = 0$: empty

$\lambda = \{2,3\}$: $\begin{pmatrix} a & 1 & 0 & c \\ b & 0 & 1 & d \end{pmatrix}$: $-b = 0$ smooth

Michelle Snider:

Showed: Given something like $4\,0\,3\,4\,0\,3\,3 \in Gr(3,7)$, one only needs to consider the maximal minors starting from each $f(i)-1<k$

~2:15

Stratification: collection of subvarieties such that the intersection of any 2 is a union of others

~~The positroid stratification~~ Then can talk about stratification generated by some subvarieties

~~The~~ intersect and decompose, these also have to be in anything containing these subvarieties

Intersect & decompose, intersect & decompose... → will be closed under intersect & decompose – so will be stratification

KLS: the positroid stratification is generated by divisors:

have n codim-1 positroid varieties that say these k columns are of rank at most k-1

from side swaps they are $k\,k \cdots k\,k{+}1\,k{-}1\,k \cdots k$ → n of these by cyclic rotation

In some sense whole stratification is controlled by these

Say we have 2 stratified spaces and want to show a map between them is a isomorphism of stratified spaces

then can do that by checking on both sides that this limited system of strata generates the stratification, and the correspondence preserves these strata.

That's what Michelle does: if I want to show there is a stratification-preserving isomorphism between this open cell of a Grassmannian and this opposite Bruhat cell in an affine flag variety

then it's enough to check that this taking divisors to divisors

Pipe dream story:

Degeneration of matrix Schubert variety $\overline{X}_\pi$ to $\operatorname{Init} \overline{X}_\pi$

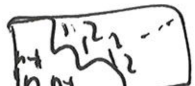
→ is $\overline{X}_{123\cdots r+1, r, \cdots n}$ — These are the divisors

So get stratification by intersect decompose until you have empty →

corresponds to doing Grobner basis calculation on the ideal (algebra) side

We are doing some sort of limit (degeneration?) that only keeps the lexicographically minimal

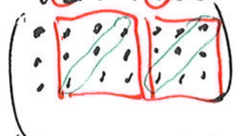
things → these are the antidiagonals (in Michelle's ordering?)

See also Snider Lemma 8. With this new order ,

$$\operatorname{Init}\left(\prod_{i=1}^{n} \det M_{[i,i+k-1]}\right) = \prod_{i,j} a_{ij}$$

smooth Deodhar's

This all means that for smooth like $4\,2\,3\,4\,2\,3\,3$, we only need to set the $\boxed{\text{determinants}}$ equal to zero



and in fact, only the anti diagonal terms equal to zero

In general since this is $abc=0 \iff a=0$ or $b=0$ or $c=0$

these are crosses in the pipe dreams.

I took this idea simply of taking determinants to be zero, and the fact that they have to be linear, to say that it requires a way of filling these "boxes" $\cdot\cdot0|\cdot\cdot0$ with columns x, such that their determinants are equal to zero.

Since we will get the pipe dreams by putting crosses where the ~~diagonals~~ zeros are

smooth $\iff$ linear $\iff$ single pipe dream

(means antidiagonal is just one term $a_{ij}=0$)

Going back to our torus invariant subset thing: This says it's smooth iff it's a single torus-invariant subset $\begin{pmatrix} 0 & x & xx & x \\ x & x & 0 & \end{pmatrix}$

Torus invariant varieties with a smooth point

If we represent elements of the Grassmannian $Gr(k,n)$ by $k \times n$ matrices, then the torus action acts by scaling columns:

$$\begin{array}{cc} & n \\ k & \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \end{array} \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \quad z_i \in \mathbb{C}^* \quad = \begin{bmatrix} z_1 v_1 & \cdots & z_n v_n \end{bmatrix}$$

so the torus fixed points are the k-planes that do not change when you scale any of the coordinates $\Rightarrow$ this means that the plane must be spanned by coordinate vectors.
So the torus fixed points are the coordinate k-planes: there are $\binom{n}{k}$ of them.

These points look like
$$\begin{bmatrix} & 1 & & 0 & & 0 & \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ & 0 & & 0 & & 1 & \\ & \vdots & & \vdots & & \vdots & \\ & 0 & & 0 & & 0 & \end{bmatrix}$$
k of these, denote this set $\lambda \in \binom{[n]}{k}$

And these points are each contained in the natural T-invariant open set:

$$\cancel{GL(k)} \begin{bmatrix} * & * & 1 & * & 0 & \cdots & 0 & * & * \\ * & * & 0 & * & 1 & \cdots & 0 & * & * \\ * & * & 0 & * & 0 & \cdots & \vdots & * & * \\ \vdots & & \vdots & \vdots & \vdots & & 0 & & \vdots \\ * & * & 0 & * & 0 & \cdots & 1 & * & * \end{bmatrix}$$
where the $*$ are free to vary

We will call these open sets U_λ corresponding to the choice of λ

U_λ is just the standard affine open cover of the Grassmannian.

But now we're considering not just the Grassmannian, but with a torus action.
Clearly if we scale this open set by the torus action, we get something else in here, so it is torus invariant. But maybe it's not so clear since we should really unscale for the ~~relevant~~ coordinate columns, and if we see how that's done it will show us how to do better and get all torus-invariant subvarieties with a smooth point on them.

So $$\begin{bmatrix} ** & 1 & * & 0 & & 0 & * \\ \vdots & 0 & * & 1 & & \vdots & \vdots \\ \vdots & * & * & 0 & \cdots & \vdots & \vdots \\ & & & \vdots & & 0 & \\ ** & 0 & * & 0 & & 1 & * \end{bmatrix} \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} = \begin{bmatrix} z_1\begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} & z_2\begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} & \begin{matrix} z_3 \\ 0 \\ \vdots \\ 0 \end{matrix} & z_4\begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} & \begin{matrix} 0 \\ z_5 \\ 0 \\ \vdots \\ 0 \end{matrix} & \cdots \end{bmatrix}$$

But we want to stay in the affine open set where the columns λ are the basis vectors.
So we have to act with $\begin{pmatrix} z_3^{-1} & & \\ & z_5^{-1} & \\ & & \ddots \end{pmatrix} \in GL(k)$ on the left

and we will get:

$$\begin{bmatrix} \frac{z_1}{z_3}* & \frac{z_2}{z_3}* & 1 & \frac{z_4}{z_3}* & 0 & & \\ & & 0 & & 1 & & \\ \frac{z_1}{z_5}* & \frac{z_2}{z_5}* & \vdots & \frac{z_4}{z_5}* & 0 & \ddots & \\ \vdots & \vdots & 0 & \vdots & \vdots & & \end{bmatrix}$$

So it's clear that if we have a T-invariant subvariety with a smooth point (which we set equal to λ by $GL(k)$ on the left), ~~that~~ the scaling means that there is a different weight on each vector. Therefore, for T-invariant subvarieties with this smooth point, there are exactly $2^{k(n-k)}$ of them: of the form:

$$\begin{bmatrix} * & \square & 1 & * & 0 & \\ * & * & 0 & * & 1 & \\ \square & \square & 0 & \square & 0 & \cdots \\ * & * & \vdots & \square & \vdots & \\ \vdots & \vdots & 0 & * & 0 & \end{bmatrix}$$

Where we have used boxes to differentiate from the zeros in the λ columns.
The boxes denote zeros and the $*$ are free to vary. We can put any combination of boxes and $*$'s in the columns other than λ. These are all the subvarieties because the different weights of the torus action mean that a torus invariant subvariety must either include an entry or it must be zero.

In particular, since positroid varieties are defined by cyclic rank conditions and ranks are not affected by scaling columns by nonzero numbers, positroid varieties are of this form.

For example, in the siteswap, $ab \cdots 0 \cdots c$ the zero means that this column is zero,

so $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \vdots \\ \vdots & 0 & 0 \\ 0 & 0 \cdots & 1 \end{pmatrix}$ or if $ab \cdots n \cdots c$ the n means it's linearly independent

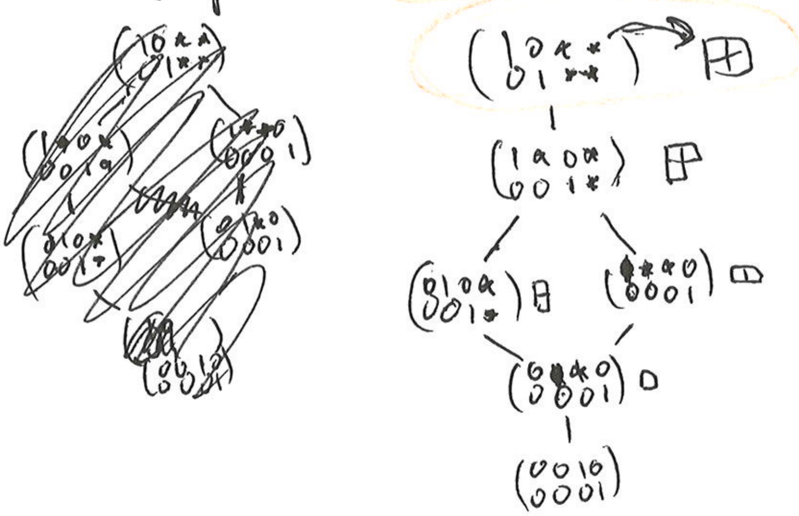
from the rest so we must have $\begin{pmatrix} & \overset{n}{0} & \\ & \vdots & \\ 0 \cdots & 0\,1\,0 & \cdots 0 \\ & \vdots & \\ & 0 & \end{pmatrix}$ ← which is also setting entries

equal to zero (and both put constraints on what columns the λ can be: they ~~...~~ must be on the n's and cannot be on the 0's).

~~...~~ Another example of something of this form is Schubert varieties, which if you've seen the Schubert decomposition, certainly looks like $\begin{pmatrix} 1 & & 0 \\ 0 & & 0 \\ 0 & & 1 \end{pmatrix}$ with 0's and $*$'s elsewhere

But furthermore, must be Torus invariant since it is also defined by rank conditions — with intervals starting from the left. Let's study what this means in the context of Schubert varieties since this will elucidate the positroid case.

Recall the usual Schubert ~~de~~decomposition of $Gr(2,4)$



– these are clearly of the torus invariant form we saw above.

How do we think of these?

$$\begin{pmatrix} 1 & \\ \vdots & \\ & \end{pmatrix}$$

look at first column, if it adds a rank (nonzero), then use $GL(k)$ to make it $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ Now look at second column: either linearly dependent so of the form $\begin{pmatrix} 1 & * & \cdots \\ 0 & 0 & \\ 0 & 0 & \end{pmatrix}$

or linearly independent so use $GL(k)$ to get $\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ 0 & 0 & \end{pmatrix}$

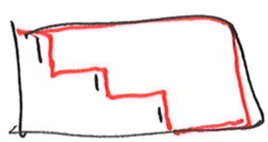
suppose latter so $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

If 3rd column is linearly dependent, then it's a combo of the first 2, so of the form $\begin{pmatrix} 1 & 0 & * & \cdots \\ 0 & 1 & * & \\ 0 & 0 & 0 & \end{pmatrix}$

and so on

We see that we have the $*$'s in the non λ-columns, so miracle $\mathbb{C}^{k(n-k)}$ – in particular forming an (inverted) Young diagram.

It is made by drawing the staircase



So we have 2 sets of numbers:

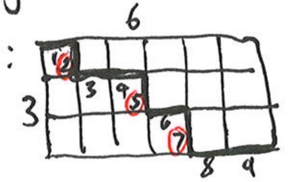
① The columns λ that are the pivots

② The number of boxes in the Young diagram we'll call $Y = \{y_1, y_2, \dots, y_k\}$

They are related by a formula you might see quite a bit:

$$\lambda_j = n-k+j-y_j$$

Here's the mnemonic:



If $(k,n)=(3,9)$

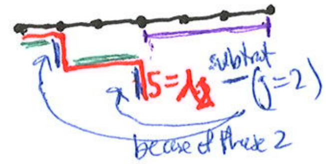
$\lambda=(5,3,2) \longleftrightarrow Y=(2,5,7)$

The drops occur along the λ columns circled

So we should think of it as: ~~[illegible]~~

$$y_j = \underset{6}{\underset{\shortparallel}{(n-k)}} - (\lambda_j - j)$$

e.g. for $j=2$



$5=\lambda_2$ subtract $-(j=2)$ = left over horizontal pieces

because of these 2

so if you subtract the green from $n-k$ you're left with the # boxes to the right = Young diagram in 2nd row

Note that sometimes you see people talk about something like

"planes Λ such that intersection with V_i has dimension $\geq$ #"

This is not the way I think about it.

But for completeness, they are thinking of the coordinates starting from the right:

$V_n \supseteq \dots \supseteq V_2 \supseteq V_1$ $\begin{bmatrix} & & \\ & & \end{bmatrix}$ and a generic k-plane won't intersect V_i until $i=n-k$

Indeed the "top-cell" generic thing $\begin{pmatrix} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{pmatrix}$ the first time you get a vector in V_i is when at the $(n-k)^{th}$

So you can think of the "number of steps early" in the intersection as the number of extra zeroes on the left: so for example for the most non-generic thing: the k-plane that is equal to V_k itself: $\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & & 0 & 0 & 1 & 0 \\ 0 & \cdots & & 0 & 0 & 0 & 1 \end{pmatrix}$, there are an extra $(n-k)$ zeros on the left of each row:

The plane jumps up in dimension $+1, +2, \dots, +k$ (for its intersection with the flag/coordinates) $n-k$ steps early for all of these k steps

Here's the important thing to note:

The Schubert variety is defined by rank conditions from the left:

$$\begin{bmatrix} 1 & \alpha & 0 & * & * & 0 & \\ 0 & 0 & 1 & * & * & 0 & \\ 0 & \vdots & 0 & 0 & 0 & 1 & \cdots \\ 0 & \vdots & \vdots & \vdots & \vdots & 0 & \\ 0 & 0 & 0 & 0 & 0 & \vdots & \end{bmatrix}$$ says $\operatorname{rank}[1,2] \le 1$, $\operatorname{rank}[1,5] \le 2$, $\cdots$ etc.

Since we saw already that positroid varieties are defined by all cyclic rank conditions, we can think of positroid varieties as ~~the~~ finer decomposition of the Grassmannian where we can rotate the basis, e.g.

$$\begin{pmatrix} * & * & * & * & 0 & * & 1 & * & * & 0 & * & * \\ & & & \vdots & * & 0 & & & & 1 & * & * \\ - & \cdots & & \vdots & * & \vdots & & & & 0 & \cdots & \\ & & & 0 & * & 0 & & & & \vdots & & \\ & & & 1 & * & & & & & 0 & & \end{pmatrix}$$

This is basically Suho Oh's paper showing that positroids are exactly intersections of cyclically rotated Schubert matroids.

Schubert matroids are the matroids (in fact positroids) of these Schubert cells:

Clearly the matroid of something like $\begin{pmatrix} 1 & * & 0 & * & * & 0 & 0 & * \\ 0 & 0 & 1 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$ is all $I \ge \{1,3,6,7\}$ in the Gale order

Actually this is Postnikov's theorem 17.2.

For a positroid cell S_M^{tnn}, $\mathcal{I}_M = (I_1, \dots, I_n)$ the Grassmann necklace corresponding to M

$$S_M^{tnn} = \bigcap_{i=1}^{n} \Omega_{I_i}^{i} \cap Gr_{kn}^{tnn}$$

These are exactly

This shows that positroids are among the intersections of cyclically shifted Schubert matroids (but not necessarily equal because of the intersection with Gr_{kn}^{tnn} might eliminate some of the matroids).

But Suho Oh proved they are equal:

<u>Thm</u> M is a positroid iff for some Grassmann necklace $(I_1, \dots, I_n)$,

$$M = \bigcap_{i=1}^{n} SM_{I_i}^{i}, \quad \text{where } SM_I^{t} := \left\{ J \in \binom{[n]}{k} \,\middle|\, I \le_t J \right\}$$

In other words, M is a positroid iff the following holds:

$H \in M$ iff $H \ge_t I_t$ for all $t \in [n]$

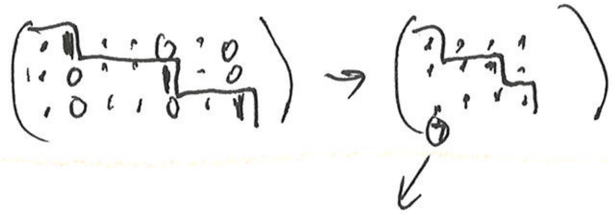
Turns $6 \subseteq$ Schubert?

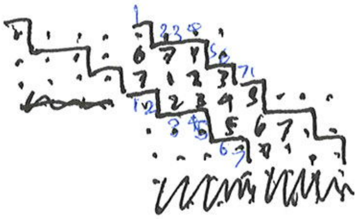
Final Example

Take 2225379 ∈ Gr(4,7)

1	2	3	4	5	6	7
3	4	5	9	8	13	14

Take λ = 2, 5, 7





Makes sense since 1 ← a cross here would switch 1 and 2

	1	2	3	4	5	6	7	8
	3	4	5	9	8	13	14	
4	3	4	5	8	9	13	14	10
47	7	4	5	8	9	13	10	14
476	7	4	5	8	9	10	13	
4761	4	7	5	8	9	10	13	11
47617	6	7	5	8	9	10	11	13
476172	6	5	7	8	9	10	11	
4761721	5	6	7	8	9	10	11	

1	2	3	4	5	6	7	8	
3	4	5	9	8	13	14		
3	4	5	8	9	13	14	10	4
7	4	5	8	9	13	10	14	47
4	7	5	8	9	13	10	11	471
4	5	7	8	9	13	10	11	472
4	5	7	8	9	10	13	11	4726
6	5	7	8	9	10	11	13	47267
5	6	7	8	9	10	11	12	472671

If you don't get the +k onesunp, then it's empty

Note that we read

this way

We could have actually gotten the bottom pipe dream

doing

To get top pipe dream:

① Shift the ordering to the top line

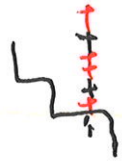
② Compute the inverse permutation (here: 4255500)

③ Go in reverse order:

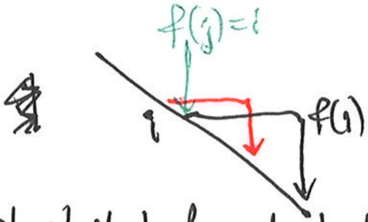
Smooth iff equal

Final 1

What happens when we delete/contract?

We simply, for deletion on i, take  and fill it in with crosses.

Proof: Recall .

What we showed there was that for  when we are deleting, we find the nearest path that start before and lands before, but lands after $f(j)=i$. Since we assume we have the bottom pipe down and are reading from bottom up,

the first time this happens is 

first one (closest to i) that starts before and lands before (and the fact that it's along this vertical thing means that it lands after $f(j)=i$)

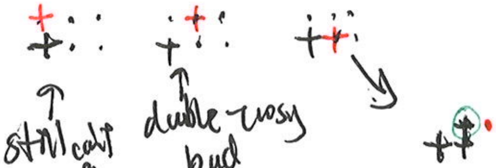
Analogously for contraction.

Here's one thing that's nice: it shows us why deletion/contraction can make something smooth but never vice-versa:

clearly if we have +· we have a move to ○. But if we contract/delete on it, we get ‡ which no longer has any moves.

But we will never get the reverse. WLOG assume for contraction we have locally something with a cross + that can't move – but we want it to be able to move after adding crosses: so ⟶ ‡ You'll never get a cross to move by adding crosses in its column → + → ‡ so assume we have + and it can't move, but you add ‡ and it can move. Since it couldn't move before we must have

Fund 2



still can't move after already crosses

double crossing bad

so you say fully in the green place it a move
But then it could have already moved to the red in the first place so it was already singular before

We also see how we can get the poset ordering:

$f \leq g$ if f can be obtained from g by putting crosses in rows or columns